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ANALYTICAL CALCULATION OF A BEAM BASED ON AN ELASTIC WINKLER FOUNDATION WITH RANGE INHOMOGENITY

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Abstract: The aim of the study is the further development of analytical methods for calculating the bending of beams resting on a non-homogeneous continuous Winkler elastic foundation. This paper considers the case when the beam is under the influence of a uniformly distributed constant transverse load, and the inhomogeneity of the elastic foundation is given by a power function with an arbitrary non-negative power exponent $m \geq 0$. Fundamental functions and a partial solution of the corresponding differential equation of beam bending are found in an explicit closed form. These functions are dimensionless and are represented by absolutely and uniformly convergent power series. In turn, the formulas for the parameters of the stress-strain state of the beam – deflection, angle of rotation, bending moment and transverse force – are expressed through the indicated functions. The unknown constants of integration in these formulas are expressed in terms of the initial parameters, which are after the implementation of the specified boundary conditions. Thus, the calculation of the beam for bending is reduced to the procedure of numerical implementation of explicit analytical formulas for the parameters of the stress-strain state.

An example demonstrates the practical application of the obtained solutions. A prismatic concrete beam based on a cubic variable elastic foundation is considered. This case corresponds to the power value $m = 3$. The results of the calculation by the author's method are presented in numerical and graphical formats for the case when the left end of the beam is hinged and the right end is clamped. The numerical values obtained by the author's method are accurate, since the applied calculation method is based on the exact solution of the corresponding differential equation. The availability of such solutions makes it possible to evaluate the accuracy of solutions obtained using various approximate methods by comparison. For the purpose of such a comparison, the paper presents the calculation results obtained by the finite element method (FEM). The absolute error of the FEM method when calculating this design was determined.

Keywords: beam, inhomogeneous elastic foundation, power-law inhomogeneity, exact solution, analytical calculation.

АНАЛІТИЧНИЙ РОЗРАХУНОК БАЛКИ, ЩО ОПИРАЄТЬСЯ НА ПРУЖНУ ОСНОВУ ВІНКЛЕРА ЗІ СТЕПЕНЕВОЮ НЕОДНОРІДНІСТЮ

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Анотація: Ціллю дослідження є подальший розвиток аналітичних методів розрахунку на згин балок, що опираються на неоднорідну суцільну пружну основу Вінклера. У даній роботі розглядається випадок, коли балка знаходиться під впливом рівномірно розподіленого сталого поперечного навантаження, а неоднорідність пружної основи задається степеневою функцією з довільним невід'ємним показником степеня $m \geq 0$. В явній замкнутій формі знайдені фундаментальні функції та частинний розв'язок відповідного диференціального рівняння згину балки. Дані функції є безрозмірними та представляються абсолютно і рівномірно збіжними степеневими рядами. В свою чергу, через вказані функції виражаються формули для параметрів напружено-деформованого стану балки – прогину, кута повороту, згинального моменту та поперечної сили. Невідомі константи інтегрування в цих формулах виражені через початкові параметри, які знаходяться після реалізації заданих граничних умов. Тим самим, розрахунок балки на згин зводиться до процедури чисельної реалізації явних аналітичних формул для параметрів напружено-деформованого стану.

На прикладі продемонстровано практичне застосування отриманих розв'язків. Розглянуто призматичну бетонну балку, що опирається на кубічно-змінну пружну основу. Такому випадку відповідає значення степеню $m = 3$. Результати розрахунку авторським методом представлені в чисельному та графічному форматах для випадку, коли лівий кінець балки вільний, а правий затиснутий. Отримані авторським методом чисельні значення є точними, оскільки застосований метод розрахунку ґрунтується на точному розв'язку відповідного диференціального рівняння. Наявність таких розв'язків дозволяє шляхом порівняння оцінювати точність розв'язків, отриманих за допомогою різного роду наближених методів. З метою такого порівняння, в роботі надано результати розрахунку, що отримані методом скінченних елементів (МСЕ). Визначено абсолютну похибку методу МСЕ при розрахунку даної конструкції.

Ключові слова: балка, неоднорідна пружна основа, степенева неоднорідність, точний розв'язок, аналітичний розрахунок.

1 INTRODUCTION

The structure, which is a beam on an elastic foundation, is often used in engineering practice, including in industrial and civil construction, in the railway industry, hydraulic engineering, shipbuilding, aerospace engineering and others.

Among the existing models of the elastic foundation, the so-called Winkler model has become widespread. In this model, the elastic foundation on which the structure rests is represented as a set of vertical, closely spaced, unrelated springs. Such a situation can generally be described by a single parameter, which is called the modulus of elasticity of the foundation or the coefficient of subgrade reaction. In the simplest case, when the elastic foundation is assumed to be homogeneous, the coefficient of subgrade reaction is constant, which significantly simplifies the solution of the corresponding differential equation of beam bending. This can explain the widely used assumption about the foundation homogeneity. However, it is common knowledge that such an assumption is far from reality and for more qualitative research it is necessary to take into account the foundation heterogeneity (variability) [1]. In this case, the coefficient of subgrade reaction will be variable along the axis of the beam, which significantly complicates the solution of the corresponding differential equation. Therefore, various approximate methods are often used to solve the problem in similar situations.

This work is devoted to the analytical calculation of the beam for bending in the case when the inhomogeneity of the elastic foundation is described by a power-law function.

2 LITERATURE REVIEW AND PROBLEM STATEMENT

Despite the large number of publications devoted to the calculation of beams on an elastic foundation, only a small number of them are devoted to the case of a variable coefficient of subgrade reaction. For the first time, such a case was presented in a monograph [2]. The author of the monograph considered a uniform beam on an elastic foundation with a linearly variable coefficient of subgrade reaction and obtained the corresponding solution based on the theory of Taylor series. Article [3] considered the case when the coefficient of subgrade reaction is a power function of the coordinate. However, the analytical solution is obtained only for the case when the degree is equal to 1. For other positive values of the degree, a numerical solution method has been developed. The authors [4] proposed an analytical method for calculating beams on heterogeneous soils, which is accompanied by a corresponding numerical scheme. The theory of Green's functions was used as a research toolkit, which made it possible to reduce the original problem to a system of integral equations. After discretization, these equations are solved numerically using the Gauss-Legendre quadrature scheme. The paper [5] considers the case of a linear variable coefficient of subgrade reaction. The analytical solution of the corresponding differential equation of beam bending is obtained here in an explicit closed form in terms of generalized hypergeometric functions. The publication [6] is devoted to thin beams on an inhomogeneous Winkler foundation. The finite difference method is used here to solve the corresponding differential equation. Using the method of homotopy analysis, the authors of [7] obtained new analytical solutions for the static deflection of anisotropic composite beams based on an elastic foundation of variable stiffness. In particular, the case where the coefficient of subgrade reaction changes according to a linear law is considered.

A detailed review of works on the bending of beams based on an elastic Winkler foundation is given in publications [8, 9]. The authors of [8], after the relevant analysis, state that there are no works in scientific periodicals that relate to analytical solutions of the beam bending problem, when the coefficient of subgrade reaction is variable, except for cases when it is given by a linear function. In the same work, the authors obtained an analytical solution with a non-linear variable coefficient of subgrade reaction. The specified solution was found for the special case when the coefficient of subgrade reaction changes according to the binomial law with the power of -4. The authors of [9], after reviewing the publications, come to the conclusion that in the case of a variable coefficient of subgrade reaction, researchers most often use FEM.

In the opinion of the authors of this work, the situation with the state of development of analytical methods for the calculation of beams based on a non-uniform elastic foundation may change significantly after the publication of the work in 2021 [10]. This assumption is based on the fact that the specified work found an exact solution to the differential equation of beam bending

$$EI\, y''''(x) + k(x)y(x) = q(x)\,, \tag{1}$$

when the linear coefficient of subgrade reaction $k(x)$ and the load $q(x)$ are given by arbitrary continuous functions and the bending stiffness EI is assumed to be constant. In the same work, an analytical method of numerical implementation of the found exact solution is proposed. Therefore, the logical continuation can be the following studies on the analytical calculation of real beam structures, which will use the results of the publication [10]. This article is an example of such research.

In [10], for functions $k(x)$ and $q(x)$ accepted representations:

$$k(x) = k_0 B(x);\;\; q(x) = q_0 C(x)\,, \tag{2}$$

where k_0, q_0 – the value of the coefficient of subgrade reaction and the load at a certain characteristic point of the beam, respectively; $B(x)$, $C(x)$ – dimensionless continuous functions, which respectively express the laws of change of coefficient of subgrade reaction and load along the beam length. In fact, $B(x)$ function is characterized by the foundation heterogeneity.

This work is devoted to the problem of analytical calculation of a beam, when the inhomogeneity of a continuous Winkler elastic foundation is expressed by a power function

$$B(x) = \left(\frac{x}{l}\right)^m,\;\; m \geq 0\,. \tag{3}$$

It is possible to consider the transverse load acting on the beam to be uniformly distributed with intensity q (Fig. 1).

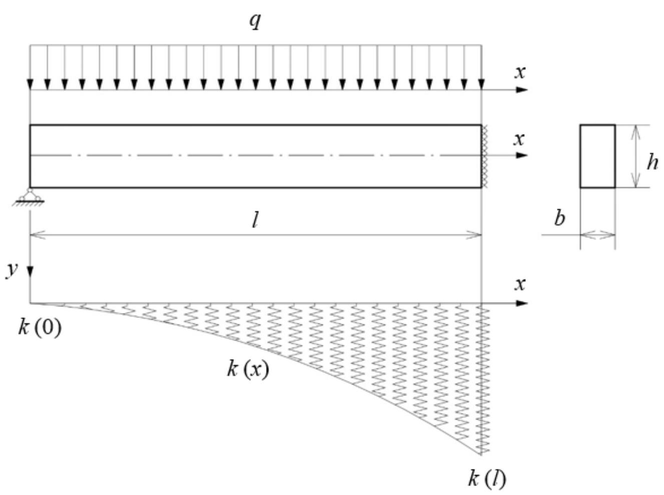


Fig. 1. Calculation scheme of the beam

In this case, according to formulas (2): $q = q_0$; $C(x) \equiv 1$; $k_0 = k(l)$.

3 THE AIM AND OBJECTIVES OF RESEARCH

The objectives of research:

1. Obtain the calculation formulas for case (3) in an explicit form;
2. Perform an analytical calculation of a real beam structure based on a non-homogeneous power-variable elastic foundation, using the author's method and FEM;
3. Determine the FEM error when calculating this design.

The aim of research is the further development of analytical methods for calculating beams based on a non-homogeneous continuous Winkler elastic foundation.

4 RESEARCH RESULTS

4.1. Calculation formulas

In the publication [10], the following general formulas were obtained for the parameters of the stress-strain state of the beam – deflection $y(x)$, angle of rotation $\varphi(x)$, bending moment $M(x)$ and transverse force $Q(x)$:

$$y(x) = y(0)X_1(x) + \varphi(0)lX_2(x) - M(0)\frac{l^2}{EI}X_3(x) - Q(0)\frac{l^3}{EI}X_4(x) + \frac{q_0 l^4}{EI}X_5(x); \tag{4}$$

$$\varphi(x) = y(0)\frac{1}{l}\tilde{X}_1(x) + \varphi(0)\tilde{X}_2(x) - M(0)\frac{l}{EI}\tilde{X}_3(x) - Q(0)\frac{l^2}{EI}\tilde{X}_4(x) + \frac{q_0 l^3}{EI}\tilde{X}_5(x); \tag{5}$$

$$M(x) = -y(0)\frac{EI}{l^2}\widehat{X}_1(x) - \varphi(0)\frac{EI}{l}\widehat{X}_2(x) + M(0)\widehat{X}_3(x) + Q(0)l\widehat{X}_4(x) - q_0 l^2\widehat{X}_5(x); \tag{6}$$

$$Q(x) = -y(0)\frac{EI}{l^3}\hat{X}_1(x) - \varphi(0)\frac{EI}{l^2}\hat{X}_2(x) + M(0)\frac{1}{l}\hat{X}_3(x) + Q(0)\hat{X}_4(x) - q_0 l\hat{X}_5(x), \tag{7}$$

where $X_n(x)$ $(n = 1, 2, 3, 4)$ – dimensionless fundamental functions of the homogeneous equation, i.e. solutions of equation (1) with zero right-hand side, $X_5(x)$ – dimensionless function through which the partial solution

$$X^*(x) = \frac{q_0 l^4}{EI}X_5(x)$$

of inhomogeneous equation (1) is expressed,

$$\tilde{X}_n(x) = lX_n'(x);\ \widehat{X}_n(x) = l^2 X_n''(x);\ \hat{X}_n(x) = l^3 X_n'''(x)\ (n = 1, 2, 3, 4, 5). \tag{8}$$

Functions $X_n(x)$ $(n = 1, 2, 3, 4, 5)$ are represented by uniformly convergent power series of a dimensionless parameter $-K$ with variable coefficients

$$X_n(x) = \beta_{n,0}(x) - K\beta_{n,1}(x) + K^2\beta_{n,2}(x) - K^3\beta_{n,3}(x) + \dots\ (n = 1, 2, 3, 4, 5), \tag{9}$$

where

$$K = \frac{k_0 l^4}{EI}.$$

Here, the initial $\beta_{n,0}(x)$ and generating $\beta_{n,k}(x)\ (k=1,2,3,...)$ functions are determined by recurrent relations, which in our case take the form:

$$\beta_{n,0}(x)=\frac{1}{(n-1)!}\left(\frac{x}{l}\right)^{n-1}\ (n=1,2,3,4,5)\,; \tag{10}$$

$$\beta_{n,k}(x)=\frac{1}{l^4}\int_0^x\int_0^x\int_0^x\int_0^x\left(\frac{x}{l}\right)^{m}\beta_{n,k-1}(x)\,dxdxdxdx\ \ (n=1,2,3,4,5)(k=1,2,3,...)\,. \tag{11}$$

By successively integrating according to formula (11), it is possible to pass from the recurrent to the analytical form of the record. As a result, for creative functions

$$\beta_{n,k}(x)=\frac{1}{(n-1)!\,p_{n,1,m}\,p_{n,2,2m}\cdots p_{n,k,km}}\left(\frac{x}{l}\right)^{n+km+4k-1}\ (n=1,2,3,4,5)(k=1,2,3,...) \tag{12}$$

where

$$p_{n,k,km}=(n+km+4k-4)(n+km+4k-3)(n+km+4k-2)(n+km+4k-1)\,.$$

Therefore, according to (9), (10), (12), the functions $X_n(x)\ (n=1,2,3,4,5)$ will be determined by the following series

$$X_n(x)=\frac{1}{(n-1)!}\left(\frac{x}{l}\right)^{n-1}\left[1+\sum_{k=1}^{\infty}\frac{(-K)^k}{p_{n,1,m}\,p_{n,2,2m}\cdots p_{n,k,km}}\left(\frac{x}{l}\right)^{k(m+4)}\right]\ (n=1,2,3,4,5)\,. \tag{13}$$

Thus, thanks to formulas (4)-(8), (13), it is possible to analytically calculate beams based on a power-variable elastic foundation with any boundary conditions.

4.2 Calculation example

It is possible to consider a prismatic concrete beam on a non-uniform elastic foundation characterized by a cubic function

$$B(x)=\left(\frac{x}{l}\right)^{3}.$$

It is possible to assume that the left end of the beam is hinged, and the right end is clamped. This method of fixing the ends of the beam will meet the boundary conditions:

$$y(0)=0;\ M(0)=0;\ y(l)=0;\ \varphi(l)=0\,.$$

So, two initial parameters $y(0), M(0)$ are known. The other two initial parameters $\varphi(0), Q(0)$ can be found from the system of equations, which will be obtained after the implementation of the boundary conditions at the end $x=l$ using formulas (4), (5). Substituting the found values for the initial parameters into formulas (4) - (7), there is:

$$y(x)=\frac{q_0\,l^4}{E_0I_0}(\lambda_1X_2(x)+\lambda_2X_4(x)+X_5(x));\ \ \varphi(x)=\frac{q_0\,l^3}{E_0I_0}(\lambda_1\tilde{X}_2(x)+\lambda_2\tilde{X}_4(x)+\tilde{X}_5(x))\,;$$

$$M(x)=-q_0l^2(\lambda_1\widehat{X}_2(x)+\lambda_2\widehat{X}_4(x)+\widehat{X}_5(x));\ \ Q(x)=-q_0l(\lambda_1\hat{X}_2(x)+\lambda_2\hat{X}_4(x)+\hat{X}_5(x))\,,$$

where

$$\lambda_1 = \frac{X_4(l)\tilde{X}_5(l) - \tilde{X}_4(l)X_5(l)}{X_2(l)\tilde{X}_4(l) - \tilde{X}_2(l)X_4(l)}; \quad \lambda_2 = -\frac{X_2(l)\tilde{X}_5(l) - \tilde{X}_2(l)X_5(l)}{X_2(l)\tilde{X}_4(l) - \tilde{X}_2(l)X_4(l)}.$$

Initial data:

$E = 1,5 \cdot 10^7\, kPa$;

Beam length $l = 5\,m$; Beam base width $b = 0,4\,m$; Beam height $h = 0,6\,m$;

$k(l) = 4 \cdot 10^3\, kN/m^2$; $q = 60\, kN/m$.

The results of the calculation by the author's method (AM) in numerical format are presented in the Tables 1 and 2, and in the graphic – in Fig. 2. For comparison in the Tables 1 and 2 also provide the results of FEM calculation in the LIRA software complex.

Table 1

Value of kinematic parameters

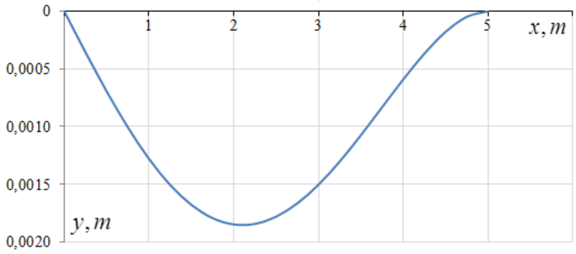
$\frac{x}{l}$	x	$y(x)$, m		Relative error, %	$\varphi(x)$, rad		Relative error, %
		AM	FEM		AM	FEM	
0	0	0,000000	0,000000	0,00	0,001430	0,001441	0,79
0,05	0,25	0,000355	0,000358	0,90	0,001399	0,001410	0,80
0,1	0,5	0,000695	0,000700	0,76	0,001312	0,001322	0,77
0,15	0,75	0,001007	0,001015	0,82	0,001178	0,001188	0,88
0,2	1	0,001280	0,001291	0,84	0,001005	0,001014	0,92
0,25	1,25	0,001507	0,001519	0,82	0,000802	0,000810	1,02
0,3	1,5	0,001679	0,001694	0,87	0,000578	0,000584	1,10
0,35	1,75	0,001794	0,001810	0,87	0,000341	0,000346	1,51
0,4	2	0,001850	0,001866	0,89	0,000100	0,000102	2,00
0,45	2,25	0,001845	0,001862	0,93	-0,000136	-0,000136	0,18
0,5	2,5	0,001782	0,001799	0,93	-0,000359	-0,000361	0,47
0,55	2,75	0,001667	0,001683	0,96	-0,000561	-0,000564	0,59
0,6	3	0,001505	0,001520	1,02	-0,000732	-0,000738	0,84
0,65	3,25	0,001304	0,001318	1,05	-0,000864	-0,000872	0,89
0,7	3,5	0,001076	0,001088	1,07	-0,000950	-0,000959	0,98
0,75	3,75	0,000834	0,000843	1,07	-0,000980	-0,000989	0,97
0,8	4	0,000592	0,000598	1,01	-0,000945	-0,000955	1,01
0,85	4,25	0,000367	0,000371	1,00	-0,000839	-0,000848	1,08
0,9	4,5	0,000179	0,000181	0,98	-0,000652	-0,000659	1,13
0,95	4,75	0,000049	0,000050	2,07	-0,000375	-0,000379	1,11
1	5	0,000000	0,000000	0,00	0,000000	0,000000	0,00

Table 2

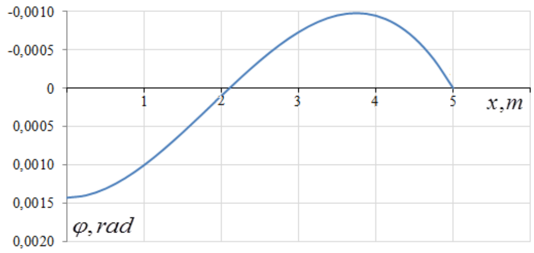
The value of power parameters

$\frac{x}{l}$	x	$M(x)$, kNm		Relative error, %	$Q(x)$, kN		Relative error, %
		AM	FEM		AM	FEM	
1	2	3	4	5	6	7	8
0	0	0,000000	0,000000	0,00	111,786398	112,266594	0,43
0,05	0,25	26,071600	26,191650	0,46	96,786409	97,266594	0,50
0,1	0,5	48,393228	48,633312	0,50	81,786748	82,266708	0,59
0,15	0,75	66,965126	67,325073	0,54	66,788992	67,267456	0,72
0,2	1	81,788193	82,267258	0,59	51,796965	52,270081	0,91
0,25	1,25	92,864630	93,460579	0,64	36,817291	37,276730	1,25

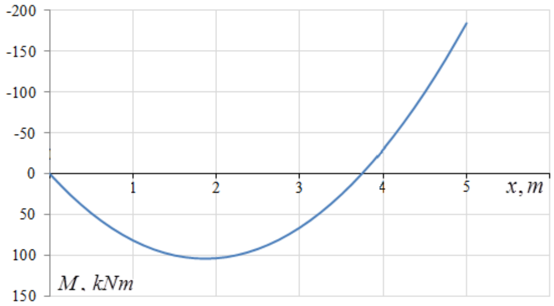
1	2	3	4	5	6	7	8
0,3	1,5	100,198646	100,906456	0,71	21,859372	22,290459	1,97
0,35	1,75	103,797088	104,607101	0,78	6,934748	7,215006	4,04
0,4	2	103,669831	104,565750	0,86	-7,944165	-7,645602	3,76
0,45	2,25	99,829783	100,786606	0,96	-22,765843	-22,587624	0,78
0,5	2,5	92,292401	93,274666	1,06	-37,521801	-37,508293	0,04
0,55	2,75	81,074645	82,035446	1,19	-52,208865	-52,406574	0,38
0,6	3	66,193380	67,074409	1,33	-66,831296	-67,283829	0,68
0,65	3,25	47,663303	48,396305	1,54	-81,402382	-82,144341	0,91
0,7	3,5	25,494580	26,004408	2,00	-95,945083	-96,995552	1,09
0,75	3,75	-0,309507	-0,303227	2,03	-110,491213	-111,847778	1,23
0,8	4	-29,754462	-29,919407	0,55	-125,078596	-126,713303	1,31
0,85	4,25	-62,855458	-63,458076	0,96	-139,745560	-141,604584	1,33
0,9	4,5	-99,636426	-100,724014	1,09	-154,522039	-156,531433	1,30
0,95	4,75	-140,126418	-141,726654	1,14	-169,416510	-171,496658	1,23
1	5	-184,352099	-186,474655	1,15	-184,397896	-186,490402	1,13



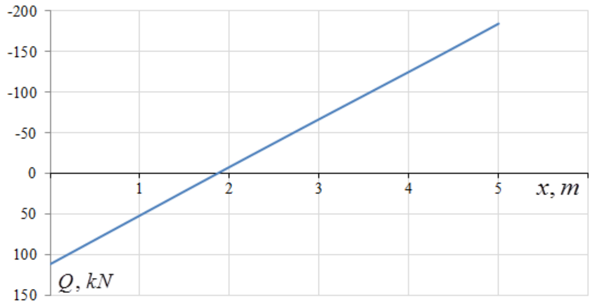
Graph of the deflection function



Graph of the turning angle function



Graph of the bending moment function



Graph of the transverse force function

Fig. 2. Graphs of beam state parameters

It should be noted that the LIRA software complex does not provide for the ability to directly specify the law of change of the coefficient of subgrade reaction along the beam length. For each finite element, the value of the coefficient of subgrade reaction is set as the arithmetic mean of the values at its ends, which affects the error value.

5 DISCUSSION OF RESEARCH RESULTS

These studies are a logical continuation of the studies initiated in the publication [10], and are entirely based on the results obtained there. In essence, it is about the application of the general solutions obtained in [10] for the case when the inhomogeneous elastic foundation is characterized by a power-law function. Since the proposed calculation method is based on the exact solution of the corresponding differential equation, it allows obtaining qualitative information and forming the most reliable picture of the stress-strain state of the beam. Corresponding numerical results obtained with the help of exact solutions are usually interpreted as exact in the scientific literature. Such solutions in closed form are particularly valuable, because they can serve as criteria by which the accuracy of various kinds of approximate solutions can be evaluated.

6 CONCLUSIONS

1. An analytical method for calculating the bending of beams based on a continuous elastic foundation with power-law heterogeneity with any power-law index is proposed $m \geq 0$.
2. The obtained numerical results of the calculation should be interpreted as accurate.
3. The error of FEM calculations in the LIRA software complex for the considered structure is determined.

7 ETHICAL DECLARATIONS

The authors have no relevant financial or non-financial interests to report.

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