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PLATE PROBLEM OF THE THEORY OF ELASTICITY FOR A COMPOSITE PLANE WITH CRACKS

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Abstract: This paper considers plane problem of linear elasticity theory for a composite plane formed by two half-planes with different elastic characteristics. It is assumed that there are cracks of finite or half-infinite length at the division line of the materials. To obtain analytical solutions, the method of integral Fourier transforms in a bipolar coordinate system is used with the application of Papkovich–Neuber solution, which allows the problems to be reduced to a closed form and their rigorous mathematical solution to be carried out.

Two characteristic cases of weakening geometry are investigated: in the first case, the composite plane is weakened by a crack of finite length located along the bimaterial interface; in the second case, two semi-infinite cracks symmetrically located about the axis which separates the half-planes are considered. In both cases, ideal (perfect) contact interaction between materials outside the crack region is assumed.

A mixed boundary value problem is solved: normal and shear stresses are specified at the crack edges, while outside the crack, continuity of displacements and stresses is ensured at the material interface, which corresponds to the condition of complete contact. The specified load functions at the edges of the crack are assumed to be piecewise smooth and satisfy the conditions of Fourier series expandability.

The distributions of normal and shear stresses along the contact line and on the crack, faces are studied under various types of external loading. Particular attention is paid to the analysis of stress behavior features in the vicinity of crack ends, where singularities of a power-law nature are observed. The results obtained can be used in the analysis of the strength and fracture of inhomogeneous materials, as well as in modeling the stress-strain state near defects at the interfaces of media.

Keywords: composite body, crack, bipolar coordinates, Papkovich–Neuber functions, Fourier transformation.

ПЛОСКА ЗАДАЧА ТЕОРІЇ ПРУЖНОСТІ ДЛЯ СКЛАДЕНОЇ ПЛОЩИНИ З ТРІЩИНАМИ

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Анотація: У цій статті розглядається плоска задача лінійної теорії пружності для складеної площини, утвореної двома півплощинами з різними пружними характеристиками. Припускається, що на лінії поділу матеріалів є тріщини скінченної або напівнескінченної довжини. Для отримання аналітичних розв'язків використовується метод інтегральних перетворень Фур'є в біполярній системі координат із застосуванням розв'язку Папковича-Нойбера, що дозволяє звести задачу до замкнутої форми та виконати її математичний розв'язок.

Досліджуються два характерні випадки геометрії ослаблення: у першому випадку складена площина ослаблена тріщиною скінченної довжини, розташованою вздовж біматеріальної межі поділу; у другому випадку розглядаються дві напівнескінченні тріщини, симетрично розташовані відносно осі, яка розділяє півплощини. В обох випадках передбачається ідеальна контактна взаємодія між матеріалами поза областю тріщини.

Розв'язується змішана крайова задача: на краях тріщини задаються нормальні та зсувні напруження, тоді як поза тріщиною забезпечується неперервність переміщень та напружень на межі поділу матеріалів, що відповідає умові повного контакту. Задані функції навантаження на краях тріщини вважаються кусочно-гладкими та задовольняють умови розкладання в ряди Фур'є.

Досліджуються розподіли нормальних та зсувних напружень вздовж лінії контакту та на границі тріщини при різних типах зовнішнього навантаження. Особлива увага приділяється аналізу особливостей поведінки напружень поблизу кінців тріщини, де спостерігаються сингулярності степеневого характеру. Отримані результати можуть бути використані при аналізі міцності та руйнування неоднорідних матеріалів, а також при моделюванні напружено-деформованого стану поблизу дефектів на межі поділу матеріалів.

Ключові слова: складене тіло, тріщина, біполярні координати, функції Папковича-Нейбера, перетворення Фур'є.

1 INTRODUCTION

Problems in elasticity theory related to the study of stress-strain states (SSS) in bodies with cracks are a significant trend in continuum mechanics and have wide application in engineering practice. Particularly topical are problems involving heterogeneous media containing defects such as cracks at the materials interface with different physical and mechanical characteristics. Such models allow adequate description of the behaviour of composite and laminated materials used in construction, aircraft engineering, mechanical engineering and other industries.

The analysis of stresses and displacements in the region of cracks at interface boundaries differs significantly from classical homogeneous problems due to the presence of elastic modulus jumps and possible peculiarities of contact interaction between material components. These peculiarities lead to the occurrence of singular stresses, which is particularly important to consider when assessing the strength and stability of structures.

This paper considers plane problems of static linear elasticity theory for a composite plane consisting of two half-planes with different elastic properties. Cracks of finite or semi-infinite length are assumed to exist at the division line between the materials. Such formulations model the most typical cases of local weakening in laminated media caused by operational damage or manufacturing defects.

The purpose of the present work is to obtain a rigorous analytical solution of the indicated problems using Fourier integral transforms in a bipolar coordinate system and the Papkovich–Neuber functions. Special attention is paid to mixed boundary conditions: stresses are specified on the crack edges, while outside the crack region ideal contact between the materials is assumed.

The solutions obtained allow not only to analyse the distribution of stresses and displacements in the contact area and near the crack ends, but also to identify the nature of singularities arising in these areas. The results presented can be used to assess the strength characteristics of multilayer structures and predict their durability.

2 ANALYSIS OF LITERARY DATA AND RESOLVING THE PROBLEM

Problems involving cracks are related to problems of determining the stress-strain state in homogeneous and inhomogeneous elastic bodies, which are of interest in both theoretical and practical issues of strength of various structures. This has become the subject of research by many authors, among whom we note [2–8, 13, 14 and the references therein].

In work [2], the concentration of elastic stresses near dies, cuts, thin inclusions is considered. Fracture of composite materials is examined in the book by Cherepanov G.P. [3]. The distribution of stresses around cracks in plates and shells is studied in the book by Panasyuk V.V. and others [4]. A mixed problem for a composite plane weakened by a crack, where stress components are specified on one crack edge and displacement components on the other, is considered in [5]. Another mixed problem for a composite plane with two semi-infinite cracks is analyzed in [6]. A contact problem for an infinite plate with a finite crack reinforced with elastic pads of the same finite length is considered in [7]. In the paper by Arutunyan L.A. [8], the elastic equilibrium of a composite plane consisting of two half-planes with semi-infinite cracks having different elastic characteristics, with stresses applied on the interface line, is studied.

Problems of torsion and bending of rods of lunular profile, bending of lunular form, as well as some plane problems for such regions were studied in works [9, 10]. In [12]

(Tarantino A.M.), the problem of plane stress state with a crack in a Mooney–Rivlin material is considered.

In our works [13, 14], plane problems for a circular segment and a half-plane with a segmental notch under mixed boundary conditions are analyzed. In a series of papers by Gao Y.C. (with co-authors) [15–23], stress analysis near the crack tip in rubber-like material was performed. Cracks in homogeneous and inhomogeneous material were considered. It was shown that the character of stress singularity depends on the elastic parameters of the material, and in the case of plate tension its thickness at the crack tip tends to zero. In article [24], where the case of an interfacial crack on the boundary of two half-planes was examined, it was established that stresses have no oscillations at the crack tip, unlike in the linear problem of the interfacial crack.

In the work of Tarantino A.M. [25], the problem of plane stress state with a crack in a Mooney–Rivlin material was studied. The equilibrium equations were written through the Airy stress functions, and approximate values were obtained by the asymptotic method. In articles [28–30], the following problems for homogeneous and two-component planes are considered: a crack in a homogeneous plane; a crack at the interface of a half-plane with a rigid element; an interfacial crack. In all cases, the generalized neo-Hookean material model was used. Comparisons were made with the results of numerical solutions by the finite element method.

An exact global solution of the nonlinear plane strain problem with an interfacial crack for a John material was obtained in the work of Malkov V.M. [31], where the Muskhelishvili complex potential method was applied. Stresses in the nonlinear interfacial crack problem have root singularity and oscillation at the crack tips, as in the linear case. In [32], an asymptotic analysis of deformations near the crack tip in a homogeneous plane for the same material model was given. The aim was to show that there exists a region where the material loses ellipticity under large deformations.

In the work of Abeyaratne R., Yang J.S. [33], stress and strain fields near the crack tip under uniaxial tension for a special type of incompressible material model were studied. It was obtained that for this model the system of nonlinear differential equations may lose ellipticity under sufficiently large deformations. The asymptotics of stresses at the tip of an interfacial crack were studied in works of Herrmann J.M. [34, 35] for generalized neo-Hookean material, with results of a large number of calculations presented.

In the paper by Akopyan V.N. (with co-authors) [36], the plane strain state of a composite elastic plane with an interfacial crack was considered, on one of the crack edges of which an absolutely rigid punch, not reaching the crack tips, is indented with adhesion.

In the present paper, two specific plane problems of the theory of elasticity are considered for a composite plane consisting of two half-planes with different elastic characteristics. It is weakened along the contact line either by one finite crack or by two semi-infinite cracks, thus transforming the domain into a doubly connected or singly connected region.

3 PURPOSE AND TAKS OF THE STADY

In the rectangular Cartesian coordinate system (x, y) the half-plane $y \geq 0$ has elastic characteristics, and the other half-plane has elastic characteristics G_1, ν_1, and the half-plane $y \leq 0$ has elastic characteristics G_2, ν_2 (G_1, G_2-shear moduli of the materials, ν_1, ν_2- Poisson's ratio).

To solve the problem, we will use the bipolar coordinate system. The relation between the rectangular coordinates (x, y)and the bipolar coordinates α, β is given by the expressions

[1, 11, 12]: $qx = sh\alpha$, $qy = \sin\beta$, $aq = ch\alpha + \cos\beta$, where α- is the dimensional parameter.

The coordinate α varies from $-\infty$ to $+\infty$. In the first half-plane, on the left $\alpha < 0$, the axis oy is the coordinate line $\alpha = 0$, points $x = \pm a$, $y = 0$ correspond to the values of $\alpha = \pm\infty$. The coordinate β varies from $-\pi$ to $+\pi$. In the upper half-plane $\beta > 0$, in the lower $\beta < 0$. The segment $(-a, a)$ is the coordinate line $\beta = 0$. As for the segment ox at $x < -a$ and $x > a$, here the coordinate β undergoes a discontinuity equal to 2π, namely on the upper bank $\beta = \pi$ and on the lower bank. $\beta = -\pi$.

The problem is solved using the Papkovich-Neuber function. According to Papkovich-Neuber, the general solution to a plane elasticity problem can be represented by three harmonic functions, since one of them is arbitrarily chosen. Taking advantage of this arbitrariness, we assume that one of the functions is identically zero.

Displacements U, V stresses and through $\sigma_x, \sigma_y, \tau_{xy}$ the Papkovich–Neuber functions are expressed as follows:

$$2GU(x,y) = \frac{\partial\Phi_0(x,y)}{\partial x} - y\frac{\partial\Phi_2(x,y)}{\partial y},$$

$$2GY(x,y) = (3-u\nu)\partial\Phi_2(x,y) + \frac{\partial\Phi_0(x,y)}{\partial y} - y\frac{\partial\Phi_2(x,y)}{\partial y},$$

$$\sigma_{x,y}(x,y) = \frac{\partial}{\partial x}\left[(1-2\nu)\Phi_2(x,y) - \frac{\partial\Phi_0(x,y)}{\partial y} - y\frac{\partial^2\Phi_2(x,y)}{\partial y^2}\right],$$

$$\sigma_x(x,y) = \frac{\partial}{\partial y}\left[2\nu\Phi_2(x,y) + \frac{\partial\Phi_0(x,y)}{\partial y}\right] - y\frac{\partial^2\Phi_2(x,y)}{\partial x^2},$$

$$\sigma_y(x,y) = \frac{\partial}{\partial y}\left[2(1-\nu)\Phi_2(x,y) - \frac{\partial\Phi_0(x,y)}{\partial y}\right] - y\frac{\partial^2\Phi_2(x,y)}{\partial y^2}, \tag{2}$$

where $\Phi_0(x,y)$ and $\Phi_2(x,y)$ the Papkovich–Neuber functions.

Statement of problem:

Let on the boundary line $y = 0$ the composite plane be weakened by a crack on the interval $|x| < a$, and a, on the semi-infinite intervals $|x| > a$ there is full contact between the materials (Fig.1).

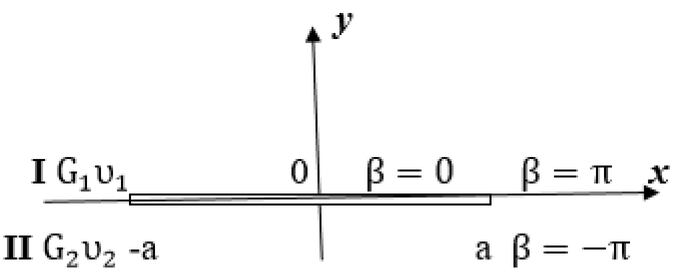

Fig. 1. *Composite plane with a finite crack*

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Let us consider a mixed boundary value problem for the given domain, when on one crack edge normal displacements and shear stresses are specified, while on the other crack edge horizontal displacement and normal stresses are specified.

$$\tau_{x,y}^{(1)}(\alpha,0)=\tau_1(\alpha),\ V_1\ (\alpha,0)=V_0(\alpha),\ \sigma_y^{(2)}(\alpha,0)=\sigma_2(\alpha),\ U_2(\alpha,0)=U_0(\alpha). \tag{3}$$

It is assumed that the functions $\tau_1(\alpha)$, $\sigma_2(\alpha), V_0(\alpha)$, $U_0(\alpha)$ satisfy the conditions of expandability into a Fourier integral. . On the contact line, full adhesion of the materials is assumed, i.e. displacements and stresses are equal:

$$\begin{gathered} U_1\ (\alpha,\pi)=U_2(\alpha,-\pi),\ V_1\ (\alpha,\pi)=V_2(\alpha,-\pi), \\ \tau_{x,y}^{(1)}(\alpha,\pi)=\tau_{x,y}^{(2)}(\alpha,-\pi),\ \sigma_y^{(1)}(\alpha,\pi)=\sigma_y^{(2)}(\alpha,-\pi). \end{gathered} \tag{4}$$

By means of expressions (2) and boundary conditions (3) and (4), through the harmonic functions $\Phi_0^{(m)}(\alpha,\beta)$, $\Phi_2^{(m)}(\alpha,\beta)$, $m=1{,}2$ they are written in the following form:

$$\begin{gathered}
\frac{\partial}{\partial\alpha}\left[(1-2\nu)\Phi_2^{(1)}(\alpha,\beta)-\Phi_3(\alpha,\beta)\right]_{\beta=0}=\frac{a\tau_1(\alpha)}{ch\alpha+1} \\
(3-4\nu_1)\Phi_2^{(1)}(\alpha,0)-\Phi_3^{(1)}(\alpha,0)=2G_1V_0(\alpha) \\
\frac{\partial}{\partial\beta}\left[2(1-\nu_2)\Phi_2^{(2)}(\alpha,\beta)-\Phi_3(\alpha,\beta)\right]_{\beta=0}=\frac{a\tau_1(\alpha)}{ch\alpha+1} \\
\left.\frac{\partial\Phi_3^2(\alpha,\beta)}{\partial\beta}\right|_{\beta=0}=2G_2\frac{\partial U_0(\alpha)}{\partial\alpha} \\
\frac{1}{G_1}\left.\frac{\partial\Phi_3^2(\alpha,\beta)}{\partial\beta}\right|_{\beta=\pi}=\frac{1}{G_2}\left.\frac{\partial\Phi_3^2(\alpha,\beta)}{\partial\beta}\right|_{\beta=-\pi} \\
\frac{1}{G_1}\left[(3-4\nu_1)\Phi_2^{(1)}(\alpha,\pi)-\Phi_3^{(1)}(\alpha,\pi)\right]=\frac{1}{G_2}\left[2(1-\nu_2)\Phi_2^{(2)}(\alpha,-\pi)-\Phi_3^2(\alpha,-\pi)\right] \\
\left.\frac{\partial}{\partial\beta}\left[2(1-\nu_1)\Phi_2^{(1)}(\alpha,\beta)-\Phi_3^1(\alpha,\beta)\right]\right|_{\beta=\pi}=\left.\frac{\partial}{\partial\beta}\left[2(1-\nu_2)\Phi_2^{(2)}(\alpha,\beta)-\Phi_3^2(\alpha,\beta)\right]\right|_{\beta=-\pi} \\
\left.\frac{\partial}{\partial\alpha}\left[(1-2\nu_1)\Phi_2^{(1)}(\alpha,\beta)-\Phi_3^1(\alpha,\beta)\right]\right|_{\beta=\pi}=\left.\frac{\partial}{\partial\alpha}\left[(1-2\nu_2)\Phi_2^{(2)}(\alpha,\beta)-\Phi_3^2(\alpha,\beta)\right]\right|_{\beta=-\pi},
\end{gathered} \tag{5}$$

where

$$\Phi_3^m(x,y)=\frac{\partial\Phi_0^m(x,y)}{\partial y},\ (m=1,2). \tag{6}$$

4 BASIC RESULTS

The harmonic functions for the first problem are sought in the form of Fourier integrals [12]

$$\Phi_3^m(\alpha,\beta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[A_n^{(m)}(\lambda)ch\lambda\beta+B_n^{(m)}(\lambda)sh\lambda\beta]\frac{\exp(-\lambda\alpha)}{\lambda}d\lambda\,. \tag{7}$$

Substituting (7) in (5) we arrive at a system of algebraic equations for determining the quantities $A_n^{(m)}(\lambda)$ and $B_n^{(m)}(\lambda)$ $(m=1,2;\ n=1,2)$, the right-hand sides of which contain the Fourier transform of the given functions.

After solving these systems, we obtain the following values for the unknown integration constants:

$$A_2^{(1)}(\lambda)=\frac{2[\bar{v}_1(\lambda)-\bar{\tau}_1(\lambda)]}{\chi_1+1},\qquad A_2^{(1)}(\lambda)=\frac{2[\bar{U}_2(\lambda)+\bar{\sigma}_2(\lambda)]}{\chi_2+1},$$

$$A_3^{(1)}(\lambda)=\frac{(\chi_1-1)\bar{v}_1(\lambda)-2\chi_1\bar{\tau}_1(\lambda)}{\chi_1+1},\qquad B_3^{(2)}(\lambda)=\bar{U}_2(\lambda),$$

$$B_2^{(1)}(\lambda)=\frac{\mu\chi_2}{\chi_1}A_2^2(\lambda)cth\lambda\pi-\frac{2\mu}{\chi_1}A_3^{(2)}(\lambda)cth2\lambda\pi+\frac{m_1(\lambda)}{\chi_1 ch\lambda\pi}+\frac{m_2(\lambda)}{\chi_1 sh\lambda\pi}, \tag{8}$$

$$B_3^{(1)}(\lambda)=-\mu A_3^2(\lambda)th\lambda\pi+\frac{m_1(\lambda)}{ch\lambda\pi},\quad A_2^{(2)}(\lambda)=\frac{\Delta_1(\lambda)}{\Delta(\lambda)},\quad A_3^{(2)}(\lambda)=\frac{\Delta_2(\lambda)}{\Delta(\lambda)},$$

where

$$\Delta(\lambda)=a_{11}(\lambda)a_{22}(\lambda)-a_{12}(\lambda)a_{21}(\lambda),$$
$$\Delta_1(\lambda)=a_{21}(\lambda)b_1(\lambda)-a_{12}(\lambda)b_2(\lambda),$$
$$\Delta_2(\lambda)=a_{11}(\lambda)b_2(\lambda)-a_{21}(\lambda)b_1(\lambda),$$

$$b_1(\lambda)=2\chi_1 m_3(\lambda)sh\lambda\pi-(\chi_1+1)m_2(\lambda)ch\lambda\pi+(\chi_1-1)m_2(\lambda)sh\lambda\pi,$$
$$b_2(\lambda)=2\chi_1 m_4(\lambda)ch\lambda\pi-(\chi_1-1)m_2(\lambda)ch\lambda\pi+(\chi_1+1)m_1(\lambda)sh\lambda\pi,$$
$$m_1(\lambda)=\mu B_3^2(\lambda)ch\lambda\pi-A_3^{(1)}(\lambda)sh\lambda\pi,$$
$$m_2(\lambda)=-\chi_1 A_2^{(2)}(\lambda)ch\lambda\pi-A_3^{(1)}(\lambda)sh\lambda\pi-\mu\chi_1 B_2^{(2)}(\lambda)sh\lambda\pi+\mu\chi_1 B_3^{(2)}(\lambda)sh\lambda\pi,$$
$$m_3(\lambda)=-\frac{\chi_1+1}{2}A_2^{(1)}(\lambda)sh\lambda\pi+A_3^{(1)}(\lambda)sh\lambda\pi+\frac{\chi_2+1}{2}B_2^{(2)}(\lambda)ch\lambda\pi-B_3^{(2)}(\lambda)ch\lambda\pi,$$
$$m_4(\lambda)=-\frac{\chi_1-1}{2}A_2^{(1)}(\lambda)ch\lambda\pi+A_3^{(1)}(\lambda)ch\lambda\pi-\frac{\chi_2-1}{2}B_2^{(2)}(\lambda)sh\lambda\pi+B_3^{(2)}(\lambda)sh\lambda\pi,$$

$$\begin{aligned}
&a_{11}(\lambda)=-\mu\chi_2(\chi_1+1)ch^2\lambda\pi+\chi_1(\chi_2+1)sh^2(\lambda\pi),\\
&a_{12}(\lambda)=-\mu_2(\chi_1+1)ch^2\lambda\pi+(\mu\chi_1-2\chi_1-\mu)sh^2\lambda\pi,\\
&a_{21}(\lambda)=[\mu\chi_2(\chi_1-1)-\chi_1(\chi_2-1)ch^2\lambda\pi,\\
&a_{22}(\lambda)=[2\chi_1-\mu(\chi_1-1)]ch^2\lambda\pi+\mu(\chi_1+1)sh^2\lambda\pi,
\end{aligned} \tag{9}$$

$$\bar{\tau}_1(\lambda)=\frac{ia}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\tau_1(\alpha)}{ch\alpha+1}\exp(i\lambda\alpha)\,d\alpha,\quad \sigma_2(\lambda)=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sigma_2(\alpha)}{ch\alpha+1}\exp(i\lambda\alpha)\,d\alpha,$$

$$\bar{v}_0(\lambda)=\frac{2G_1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}v_0(\alpha)\exp(i\lambda\alpha)\,d\alpha,\quad \sigma_2(\lambda)=\frac{2G_2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial u_0(\alpha)}{\partial\alpha}\exp(i\lambda\alpha)\,d\alpha,$$

$$\mu=\frac{G_1}{G_2},\quad \chi=3-4\upsilon_m\quad (m=1,2).$$

2. In the second variant of the considered problem, on the boundary line in the semi-infinite intervals there is a crack, while on the section full contact between the materials is assumed (Fig. 2).

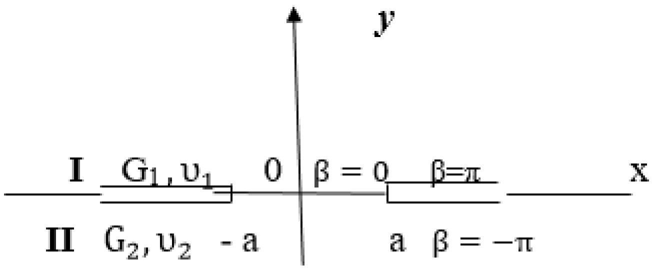

Fig. 2. *Composite plane with an infinite crack*

The boundary and contact conditions in this case have the form:

$$\tau_{x,y}^{(1)}(\alpha,\pi)=\tau_1(\alpha),\ V_1\ (\alpha,\pi)=V_0(\alpha),\ \sigma_y^{(2)}(\alpha,-\pi)=\sigma_2(\alpha),\ U_2(\alpha,-\pi)=U_0(\alpha)\,, \tag{10}$$

$$U_1(\alpha,0)=U_2(\alpha,0)\,,\ V_1\ (\alpha,0)=V_2(\alpha,0),\ \tau_{x,y}^{(1)}(\alpha,0)=\tau_{x,y}^{(2)}(\alpha,0),\ \sigma_y^{(1)}(\alpha,0)=\sigma_y^{(2)}(\alpha,0).$$

In this case, the harmonic functions $\Phi_n^{(m)}(\alpha,\beta)\ (m=1,2;\ n=1,3)$ are sought in the following form of Fourier integrals:

$$\Phi_n^m(\alpha,\beta)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[A_n^{(m)}(\lambda)ch\lambda(\pi+(-1)^m\beta)+(-1)^{m+1}B_n^{(m)}(\lambda)sh\lambda(\pi+$$

$$+(-1)^m\beta]\frac{\exp(-i\lambda\alpha)}{\lambda}d\lambda. \tag{11}$$

After satisfying the boundary and contact conditions (10), as in the first problem, for the unknown integration constant $(m=1,2;\ \ n=1,2)$ we obtain exactly the same expressions as in (8), (9).

In particular,

$$\bar{\tau}_1(\lambda)=-\frac{ia}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\tau_1(\alpha)}{ch\alpha+1}\exp(i\lambda\alpha)d\alpha\,,\ \ \sigma_2(\lambda)=\frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sigma_2(\alpha)}{ch\alpha+1}\exp(i\lambda\alpha)d\alpha \tag{12}$$

Let us consider a special case of the problems stated above, when on the lower edges of the cracks normal loads are applied, $b=ath\frac{\alpha_0}{2}$, at the point in the first case and $b=acth\frac{\alpha_0}{2}$ in the second case.

5 DISCUSSION OF THE RESULTS OF THE STUDY

We calculate the normal and shear stresses on the contact line and on the crack edges. In the first case, we obtain:

$$\sigma_y^{(1)}(\alpha,\pi)=\sigma_{\text{ъ}y}^{2}(\alpha,-\pi)=H_1[d_1k_4+(d_1+2d_2)k_3]\frac{a^2\sqrt{|x|+\sqrt{x^2-a^2}}}{2\sqrt{2}|x|}\frac{(x+a)^{2\theta}+(x-a)^{2\theta}}{(x^2-a^2)^{3/4+\theta}}\,,$$

$$\tau_{x,y}^{(1)}(\alpha,\pi)=\tau_{\text{ъ}y}^{(2)}(\alpha,-\pi)=H_1[d_3k_6+(d_3-2d_4)k_5]\frac{a^2\sqrt{|x|+\sqrt{x^2-a^2}}}{2\sqrt{2}\,|x|}\frac{(x+a)^{2\theta}+(x-a)^{2\theta}}{(x^2-a^2)^{3/4+\theta}},$$

$$\sigma_y^{(1)}(\alpha,0)=H_1[d_1k_2+d_2k_{13}]\frac{a^2}{\sqrt{2}\sqrt{a+\sqrt{a^2-x^2}}}\frac{(x+a)^{2\theta}+(x-a)^{2\theta}}{(a^2-x^2)^{3/4+\theta}},\ \ \tau_{x,y}^{(1)}(\alpha,0)=0. \tag{13}$$

For the second case, we have:

$$\sigma_y^{(1)}(\alpha,0)=\sigma_y^{(2)}(\alpha,0)=-H_1[d_1k_4+(d_1+2d_2)\kappa_3]\frac{\sqrt{a+\sqrt{a^2-x^2}}}{2\sqrt{2}}\frac{(x+a)^{2\theta}+(x-a)^{2\theta}}{(a^2-x^2)^{3/4+\theta}},$$

$$\tau_{x,y}^{(1)}(\alpha,0)=\tau_{x,y}^{(2)}(\alpha,0)=H_1[d_3k_6+(d_3-2d_4)k_5]\frac{a^2\sqrt{a+\sqrt{a^2-x^2}}}{2\sqrt{2}}\frac{(a+x)^{2\theta}+(x-a)^{2\theta}}{(a^2-x^2)^{3/4+\theta}},$$

$$\sigma_y^{(1)}(\alpha,\pi)=H_1[d_1k_2-d_2k_1]\frac{a^2}{\sqrt{2}\sqrt{|x|+\sqrt{x^2-a^2}}}\frac{(x+a)^{2\theta}+(x-a)^{2\theta}}{(x^2-a^2)^{3/4+\theta}},\ \tau_{x,y}^{(1)}(\alpha,-\pi)=0. \tag{14}$$

where

$$\alpha=\ln\left|\frac{a+x}{a-x}\right|,\ \alpha_0=\ln\left|\frac{a+b}{a-b}\right|,$$

$$\cos 4\pi\theta=k_0,\ k_0=\frac{2(\mu-1)(\mu\chi_2-1)}{(\mu+\chi_2)(\mu\chi_2+1)},\ H_1=\frac{2\mu P}{a\pi(\mu+\chi_1)(\mu\chi_2+1)},$$

$$h_1=th\alpha\theta-th\alpha_0\theta,\ h_2=1-th\alpha\theta th\alpha_0\theta,\ d_1=2\mu\chi_2+\chi_1\chi_2+1, \tag{15}$$

$$h_3=th\frac{\alpha}{4}-th\frac{\alpha_0}{4},\ h_4=1-th\frac{\alpha}{2}th\frac{\alpha_0}{2},\ d_2=\chi_1+\chi_2-2\mu\chi\chi_2,$$

$$h_5=1-th\frac{\alpha}{2}th\frac{\alpha_0}{2},\ h_6=\frac{\sqrt{2}ch\alpha_0}{4h_3\sin 4\pi\vartheta cth\frac{\alpha_0}{2}},\ d_3=\chi_1\chi_2-1,\ d_4=\chi_1-\chi_2,$$

$$k_1=\frac{h_1ch\alpha_0\theta}{4h_3\sin 4\pi\vartheta cth\frac{\alpha_0}{2}},\ k_2=\frac{h_2ch\alpha_0\theta}{4h_3\cos 2\pi\vartheta cth\frac{\alpha_0}{2}},$$

$$k_3=h_6\cos\pi\vartheta(h_1h_3-h_2h_4tg\pi\vartheta),\quad k_4=-h_6\cos 3\pi\vartheta(h_1h_3-h_2h_4tg3\pi\vartheta),$$

$$k_5=h_6\cos\pi\vartheta(h_1h_4-h_2h_3tg\pi\vartheta),\quad k_6=-h_6\cos\pi\vartheta(h_1h_4-h_2h_3tg3\pi\vartheta).$$

From here it is seen that the contact stresses at the crack tips have a power-law singularity of order $3/4+\theta$. Moreover, when $k_0\geq 0$ the order of singularity is a real number. In the case $k_0<0$ when the order of singularity is a complex number, i.e. we obtain a power-law singularity with oscillation. Note that in the case of a homogeneous plane $\theta=0$ and the contact stresses at the crack tips have a singularity of order $3/4$.

6 CONCLUSIONS

The results obtained during analytical research of plane problems of elasticity theory for a composite plane with cracks at the interface between different materials allow us to draw a number of significant conclusions regarding the nature of the stress-strain state in the contact area and near the ends of the cracks.

In a particular case, simple forms for determining the contact points of the crack have a power-law singularity of order $3/4+\theta$. Moreover, when $k_0\geq 0$ order of singularity real number. In the case when $k_0<0$ the order of the singularity is a complex number, meaning we will have a power-law singularity with oscillation. At the crack edges, the shear stress is zero.

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8 ETHICAL DECLARATIONS

The authors declare that there is no conflict of interest.

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