UDC 534:624.04

SOME PROBLEMS OF OPTIMIZATION AND CONTROL OF THE NATURAL FREQUENCIES OF AN ELASTICALLY SUPPORTED RIGID BODY

S. Bekshaev[1]
[1]*Odesa state academy of civil engineering and architecture*

Abstract. The article analytically investigates the behavior of the frequencies and modes of natural vibrations of a rigid body, based on point elastic supports, when the position of the supports changes. It is assumed that the body is in plane motion and has two degrees of freedom. A linear description of body vibrations is accepted. The problems of determining such optimal positions of elastic supports at which the fundamental frequency of the structure reaches its maximum value are considered. Two groups of problems were studied. The first group concerns a body supported by only two supports. It was found that in the absence of restrictions on the position of the supports to maximize the fundamental natural frequency, these supports should be positioned so that the basic natural vibrations of the body are translational. Simple analytical conditions are formulated that must be satisfied by the corresponding positions of the supports. In real practical situations, these positions may be unreachable due to the presence of various kinds of restrictions due to design requirements. In this paper, optimization problems are considered taking into account a number of restrictions on the position of supports, typical for practice, expressed analytically by equations and inequalities. For each of the considered types of constraints, results are obtained that determine the optimal positions of the supports and the corresponding maximum values of the main natural frequencies. The approach applied allows us to consider other types of restrictions, which are not considered in the article. In the second group of problems for a body resting on an arbitrary number of supports, the optimal position of an additional elastic support introduced in order to maximize the fundamental frequency in fixed positions and the stiffness coefficients of the remaining supports was sought. It was found that this position depends on the value of the stiffness coefficient of the introduced support. Results are obtained that qualitatively and quantitatively characterize this position and the corresponding frequencies and modes of natural oscillations, including taking into account practically established limitations. The research method uses a qualitative approach, systematically based on the well-known Rayleigh theorem on the effect of imposing constraints on the free vibrations of an elastic structure.

Keywords: vibrations, rigid body, elastic support, natural frequency, optimization.

ДЕЯКІ ЗАДАЧІ ОПТИМІЗАЦІЇ І УПРАВЛІННЯ ВЛАСНИМИ ЧАСТОТАМИ ПРУЖНО ОПЕРТОГО ТВЕРДОГО ТІЛА

Бекшаєв С.Я.[1]
[1]*Одеська державна академія будівництва та архітектури*

Анотація. У роботі аналітично досліджується поведінка частот і форм власних коливань твердого тіла, опертого на зосереджені пружні опори, за умови зміни положень цих опор. Передбачається, що тіло здійснює плоскопаралельний рух і має два ступені свободи. Приймається лінійний опис коливань тіла. Розглянуто задачі визначення таких оптимальних положень пружних опор, при яких основна частота власних коливань конструкції досягає максимального значення. Вивчено дві групи задач. Перша група стосується тіла, опертого лише на дві опори. Встановлено, що за відсутності будь-яких обмежень на положення опор для максимального підвищення основної частоти ці опори повинні розташовуватися так, щоб основні власні коливання тіла були поступальними. Сформульовано прості аналітичні умови, яким мають задовольняти відповідні положення опор. У реальних практичних ситуаціях ці

https://doi.org/10.31650/2618-0650-2021-3-2-88-102

положення можуть виявитися недосяжними у зв'язку з наявністю різноманітних обмежень, обумовлених проектними вимогами. У роботі розглянуто проблеми оптимізації з урахуванням низки характерних для практики обмежень на положення опор, які аналітично виражаються рівняннями або нерівностями. Для кожного з розглянутих видів обмежень отримано результати, які визначають оптимальні положення опор та відповідні максимальні значення основних власних частот. Застосований підхід дозволяє розглянути інші види обмежень, не розглянуті у статті. У другій групі задач для тіла, опертого на довільне число опор, розшукувалося оптимальне положення додаткової пружної опори, яка вводиться з метою максимального підвищення основної частоти при фіксованих положеннях і коефіцієнтах жорсткості інших опор. Встановлено, що це положення залежить від величини коефіцієнта жорсткості введеної опори. Отримано результати, які якісно та кількісно характеризують це положення та відповідні частоти та форми власних коливань, у тому числі з урахуванням практично обумовлених обмежень. Метод дослідження використовує якісний похід, що систематично спирається на відому теорему Релея про вплив накладання в'язі на вільні коливання пружної конструкції та її наслідки.

Ключові слова: коливання, тверде тіло, пружне закріплення, власна частота, оптимізація.

1. INTRODUCTION

During the operation of engineering structures, instruments and equipment experiencing dynamic effects, significant forces and displacements can occur in the elements of these structures, which can make it impossible to use them. If the structure contains elastic elements and is subjected to periodic loads, such conditions can be associated with the occurrence of resonance phenomena. One of the ways to eliminate these phenomena is to purposefully control the spectrum of natural frequencies of the structure by choosing its elastic-geometric characteristics. In particular, it is possible to influence this spectrum by choosing the location of the elastic constraints available to the designer. In the proposed work, some problems of controlling the natural frequency of an elastically fixed rigid body are considered. A feature of the approach used is the predominant use of qualitative methods.

2 LITERATURE ANALYSIS AND PROBLEM STATEMENT

There is a large number of studies devoted to the problems of control and optimization of the characteristics of engineering structures, in particular the frequencies of their natural vibrations [1]. Most of them use algorithms based on some of the many general mathematical optimization methods [2]. For the problems of controlling the frequency spectrum of an elastically fixed rigid body, were used numerical methods based on an enumeration of options [3]. At the same time, the solution of some problems can be found on the basis of simple qualitative considerations dating back to Rayleigh [4], without using complex mathematical methods and the associated formalization. A similar approach has been successfully applied in solving some problems of controlling the natural frequencies of elastic rod systems by varying the positions of the supports [5, 6]. In the present work, it is used to solve some problems related to the search for the positions of elastic supports of an absolutely rigid body, which provide the maximum of its fundamental natural frequency. Some results concerning the influence of the position of the supports of an elastically fixed rigid body on its natural frequencies are presented in article [7]. This work can be considered as its continuation and uses the conclusions obtained in it.

3 THE PURPOSE AND OBJECTIVES OF THE STUDY

The aim of this work is to study the behavior of the fundamental frequency of an engineering structure installed on point elastic supports when the position of the supports changes. In particular, positions are sought in which the frequency reaches its maximum, including under some limitations typical for practice. A mechanical model of a structure in the form of an absolutely rigid body, supported on linear elastic supports of finite rigidity, is used. It is assumed that it is possible to describe the model as a linear elastic system with two degrees of freedom. A similar model for the case of two supports was considered in various works, in particular in [8–10], but without any connection with optimization. As in [7], the study is based on the qualitative results of the theory of oscillations.

4 RESEARCH RESULTS

4.1. Model description. The model of the structure is considered, shown in Fig. 1 a. The support points, like the center of mass of the model, are located on one straight line, hereinafter called the axis of the model.

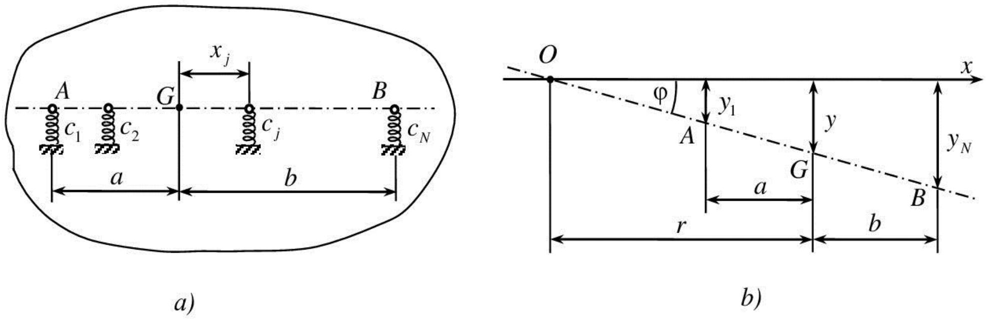

Fig. 1. The base model BM *(a)* and its coordinates *(b)*

It is assumed that during free oscillations, the displacements of the points of the model are parallel to the plane of the drawing, and the displacements of the points of the axis are perpendicular to the axis, so that the model has two degrees of freedom. The mass of the model is equal to, the moment of inertia about the axis passing through the center of mass and perpendicular to the plane of the figure is equal to.

On the model axis, we introduce the coordinate axis with the origin at its center of mass. The support numbered has a stiffness coefficient and a coordinate.

Along with the base model (BM), which has specific limited dimensions, we will also consider an extended model (EM) of the structure, which includes the points of its axis, unrestrictedly extended in both directions. The movement of the EM is completely determined by the movement of the BM.

4.2. Preliminary results. In what follows, we use the notation

$$C = \sum c_j,\ S = \sum c_j x_j,\ Q = \sum c_j x_j^2,\quad j = 1, 2, \ldots, N, \tag{1}$$

$j = 1, 2, \ldots, N$, N – number of resilient supports. In what follows, it is assumed everywhere that $S \le 0$. This can always be provided by choosing the direction of the axis x. All considerations are carried out from the point of view of an observer positioned so that the axis x is directed to the right.

The article systematically uses

4.2.1. **Rayleigh's theorem on the effect of imposition of constraint on the natural frequencies of an elastic system** [4]: Let λ_1 and λ_2 be the squares, respectively, of the first (fundamental) and second natural frequencies of the elastic structure, λ_1^* – the square of the fundamental frequency of the reinforced structure, formed from the original by imposing one constraint. Then the relations are satisfied

$$\lambda_1 \le \lambda_1^* \le \lambda_2. \tag{2}$$

Equalities in (2) can be realized only if the constraint is established at the node of the vibration mode corresponding to one of the frequencies $\sqrt{\lambda_1}$ or $\sqrt{\lambda_2}$.

The following results are also used [7].

4.2.2. Properties of frequencies and modes of free vibration of the model. The frequencies and modes (eigenforms) of free vibrations are determined by the system

$$\left.\begin{aligned}(C - M\lambda)Y + S\Phi = 0\\ SY + (Q - J\lambda)\Phi = 0\end{aligned}\right\}, \tag{3}$$

where Y and Φ are the amplitude values of the displacement y of the center of mass G and the angle φ of inclination of the model axis (Fig. 1 *b*). The eigenforms satisfy the orthogonality relation

$$MY_1Y_2 + J\Phi_1\Phi_2 = 0, \tag{4}$$

where Y_k is the vertical displacement of the center of mass, Φ_k is the angle of inclination of the axis of the *k*-th eigenform corresponding to the frequency $\sqrt{\lambda_k}$.

If, during free vibrations $\Phi \neq 0$, the corresponding mode has a node – the point of the EM axis, which remains stationary (point *O* in Fig. 1 *b*).

Relation (4) shows that for $Y_1 = Y_2 \neq 0$ Φ_1 and Φ_2 have different signs, i.e. if the center of mass is not a node of any of the modes, the nodes of two simple modes are located on opposite sides of the center of mass. It also follows from (4) that one of the two modes corresponding to simple frequencies is horizontal ($\Phi = 0$, $Y \neq 0$) if and only if the other has a node at the center of mass ($Y = 0$, $\Phi \neq 0$).

Equations (3) show that such modes can be realized if and only if $S = 0$.

As seen from Fig. 1 *b*, $Y_k = \pm r_k\Phi_k$, where r_k is the distance of the center of mass from the node of the *k*-th mode, the "+" sign is chosen if the node is located to the left of G, "–", if to the right. Then the orthogonality relation (4) can be rewritten as

$$r_1r_2 = \frac{J}{M}. \tag{5}$$

If the mode has a node, the square of the corresponding frequency is

$$\lambda = \frac{\sum c_j X_j^2}{J + Mr^2}, \tag{6}$$

where r is the distance from the node to the center of mass G, $X_j = r + x_j$ is the distance from the node to the *j*-th support.

4.2.3. Change in natural frequencies when changing the position of the supports. If the mode corresponding to a simple frequency has a node, an increase in the distance of any of the supports from the node leads to an increase in this frequency. So, it follows that extremal values of frequencies can be reached when the movable support is in a node of the corresponding mode, or this mode does not have a node (the axis is horizontal).

The following results are for a two-support model.

4.2.4. Qualitative features of eigenforms. The modes of free vibrations of a two-support model can be of three types. A mode of the 1st type has an external node located on one side of both supports, or does not have a node (horizontal). The form of the 2nd type has an internal node located between the supports. A special mode has a node located on one of the supports.

If one of the two modes of the model (as a system with two degrees of freedom) is of the 1st type, then the other is of the 2nd type. If one of the two modes of the model is special, then the other is also special with a node on the opposite support.

4.2.5. Criterion of form type. The fundamental mode is a mode of the 1st kind, if $J < Mab$, of the 2nd kind, if $J > Mab$, and special, if $J = Mab$.

4.2.6. Criterion of node position. If $c_1a > c_2b$, the fundamental mode has a node to the left of the center of mass. If $c_1a = c_2b$, one of the two modes of the model (depending on the sign of $J - Mab$) does not have a node (the axis is horizontal), and the other has a node in the center of mass.

https://doi.org/10.31650/2618-0650-2021-3-2-88-102

4.3. Optimization. The position of the supports is considered optimal if in this position the fundamental frequency of the model reaches its maximum.

4.3.1. Optimization of a two-support model, $N = 2$, $x_1 = -a$, $x_2 = b$. The values of M, J, c_1 and c_2 are specified. Optimal values of a and b are sought.

The imposition of one constraint prohibiting the rotation of the body leaves it with one degree of freedom, corresponding to the translational vertical (VT) oscillatory motion with a frequency whose square is equal to

$$\lambda_{\mathrm{VT}} = \frac{c_1 + c_2}{M} \tag{7}$$

and which does not depend on the position of the supports, $\lambda_{\mathrm{VT}}(a,b) = \mathrm{const}$. Hence, on the basis of Sec. 4.2.1, it follows $\lambda_1 \le \lambda_{\mathrm{VT}} \le \lambda_2$, i.e. $\lambda_1 = \lambda_1(a,b)$ as a function of the support positions cannot exceed λ_{VT}. In accordance with Sec. 4.2.6, translational oscillations with frequency $\sqrt{\lambda_{\mathrm{VT}}}$ will occur if and only if $c_1 a = c_2 b$. So this equality determines the optimal position of the supports, provided that the translational motion will be the fundamental one, i.e. at $J \le abM$. If the design requirements do not allow this inequality to be ensured, the fundamental vibrations will occur in the mode of the 2nd type. This can be the case if the project establishes restrictions on a and b, in particular, on the distance $\ell = a + b$ between the supports. In this case, the fundamental frequency will be less than $\sqrt{\lambda_{\mathrm{VT}}}$, but, as can be seen from the inequalities $\lambda_1 \le \lambda_{\mathrm{VT}} \le \lambda_2$, $\sqrt{\lambda_{\mathrm{VT}}}$ is the lower boundary of the second frequency of the two-support model.

To find the optimal positions of the supports, we will use the following graphical representation.

Fig. 2 shows the coordinate plane, the point of which with coordinates a and b corresponds to the position of the supports A and B. Lines are the graphs of the dependencies $b = (c_1/c_2)a$ and $b = J/Ma$. Solid lines divide the plane into three areas, the points of which cannot correspond to the optimal positions. In the area I, inequalities $c_2 b > c_1 a$ and $J < abM$ are satisfied, which, in accordance with Sec. 4.2.5 and 4.2.6, means that in these positions the fundamental mode of the EM has a node to the right of both supports (see the top row in the table on the right in Fig. 2), and in accordance with Sec. 4.2.3, the frequency can be increased by moving the supports away from the node, i.e. increasing a and decreasing b. In this case, the corresponding point of the plane moves in the directions of the "arrows of growth" shown in area I.

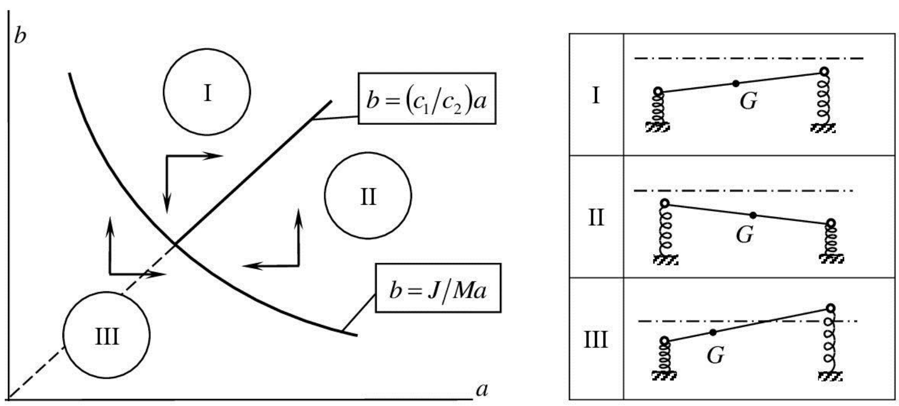

Fig. 2. Mapping a set of two-support models on a plane *ab*

In the area II, inequalities $c_2 b < c_1 a$ and $J < abM$ are satisfied, which means that in these positions the fundamental mode of EM has a node to the left of both supports (see the second row in the table on the right), and the frequency can be increased by moving the supports away from the node, i.e. decreasing a and increasing b. In this case, the corresponding point of the plane moves in the directions of the "arrows of growth" shown in area II.

Finally, in area III, the inequality $J > abM$ is fulfilled, which means that in these positions the fundamental mode of the model has a node between the supports (see the bottom row in the table on the right), and the frequency can be increased by moving the supports away from the node, i.e. increasing both a and b. In this case, the corresponding point of the plane moves in the directions of the arrows shown in the area III. The dashed line does not separate qualitatively different fundamental modes.

Thus, the maximum of the frequency of the EM can occur only in the positions corresponding to the solid lines. It should be kept in mind that the points lying on the curve $b = J/Ma$, according to Sec. 4.2.5, correspond to special modes. For the left branch, the fundamental mode has a node on the right support B $\left(X_1 = a+b,\ X_2 = 0,\ r = b\right)$ and, according to (6), the following equality is satisfied

$$\lambda_1 = \frac{\sum c_j X_j^2}{J + Mr^2} = \frac{c_1\left(a+b\right)^2 + c_2 \cdot 0^2}{J + Mb^2} = \frac{c_1\left(a+b\right)^2}{Mab + Mb^2} = \frac{c_1}{M}\left(1+\frac{a}{b}\right) = \frac{c_1}{M}\left(1+\frac{Ma^2}{J}\right) = \frac{c_1}{M}\left(1+\frac{J}{Mb^2}\right) \quad (8)$$

The right branch corresponds to the modes with a node on the left support A $\left(X_1 = 0,\ X_2 = a+b,\ r = a\right)$ and, accordingly

$$\lambda_1 = \frac{\sum c_j X_j^2}{J + Mr^2} = \frac{c_1 \cdot 0^2 + c_2\left(a+b\right)^2}{J + Ma^2} = \frac{c_2\left(a+b\right)^2}{Mab + Ma^2} = \frac{c_2}{M}\left(1+\frac{b}{a}\right) = \frac{c_2}{M}\left(1+\frac{Mb^2}{J}\right) = \frac{c_2}{M}\left(1+\frac{J}{Ma^2}\right) \quad (9)$$

The relations (8, 9) show that the fundamental frequency of the model corresponding to the curved line increases monotonically when approaching the branch point at which

$$c_1 a = c_2 b \Rightarrow J = Mab = M\left(c_2/c_1\right)b^2 = M\left(c_1/c_2\right)a^2,\ \lambda_1 = \lambda_{VT}. \quad (10)$$

Thus, the EM has infinite number of optimal positions corresponding to the points of the solid half-line. For these positions $\lambda = \lambda_{max} = \lambda_{VT}$.

In real structures, in contrast to EM, there are restrictions imposed on a and b, which separate a certain permissible area on the plane $a\,b$. To find the optimal positions, this area should be superimposed on the plane $a\,b$ with the shown solid lines. If a solid rectilinear segment falls into it, then the corresponding positions will be the solution to the problem. Otherwise, the optimal positions should be sought on the border of the permissible area, taking into account the directions of the "arrows of growth" λ (Fig. 2).

Consider the following optimization problems.

4.3.1.1. Problem 1. It is required to find the optimal positions of the supports under the given limitations

$$a \le a_1,\ b \le b_1, \quad (11)$$

where a_1 and b_1 are given constants.

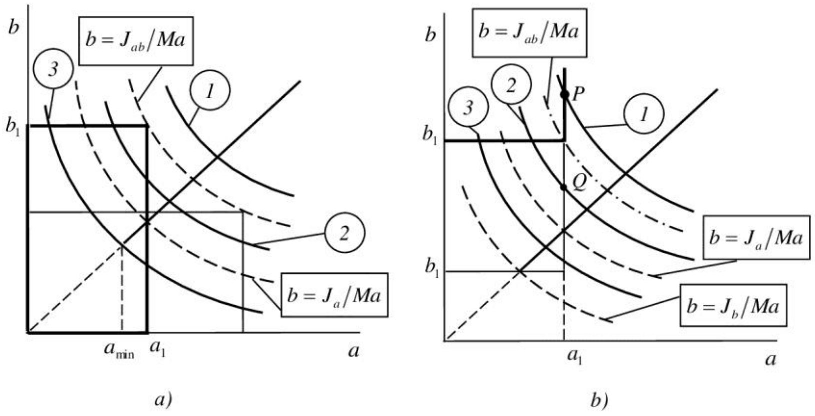

Fig. 3. To the determination of the optimal positions of the supports of the two-support model under the limitations

a) – for the case $a \le a_1$, $b \le b_1$, *b)* – for the case $a \le a_1$, $b \ge b_1$

Let us consider separately the cases $c_2 b_1 > c_1 a_1$ and $c_2 b_1 < c_1 a_1$. Let us introduce the notation

$$J_{ab} = M a_1 b_1,\; J_a = M(c_1/c_2)a_1^2,\; J_b = M(c_2/c_1)b_1^2 .$$

Let's assume at the beginning that $c_2 b_1 > c_1 a_1$ (Fig. 3 *a*). Then $J_{ab} > J_a$. Let's select three areas on the plane $a\,b$: $J \ge J_{ab}$, $J_a \le J < J_{ab}$, $J < J_a$. In fig. 3*a*, they are separated by dashed curved lines. The admissible area (11) is a rectangle with sides a_1 and b_1.

At $J \ge J_{ab}$ (line 1) none of the solid lines falls into the admissible area (point (a_1, b_1) turns out to be in the area III according to Fig. 2). Therefore, in accordance with the directions of the "arrows of growth" in Fig. 2, the optimal position is $a_{\text{opt}} = a_1$, $b_{\text{opt}} = b_1$. The fundamental mode has an inner node (is a mode of the 2nd type).

In the case $J_a \le J < J_{ab}$ (line 2), a part of the admissible area is cut off by line $b = J/Ma$, above which at $c_2 b > c_1 a$ optimal positions there cannot be (area I in Fig. 2). Therefore, in accordance with the directions of the "arrows of growth" the optimal position is $a_{\text{opt}} = a_1$, $b_{\text{opt}} = J/Ma_1$. In this case, according to (8), $\lambda = \lambda_{\max} = (c_1/M)\left(1 + Ma_1^2/J\right)$ and the fundamental mode has a node on the right support.

Finally, at $J < J_a$ (line 3), a solid rectilinear segment falls into the admissible area, the boundaries of which have coordinates $a_{\min} = \sqrt{(J/M)(c_2/c_1)}$, $a_{\max} = a_1$. In this case, there are infinite number of optimal positions of the supports, the coordinates of which are determined by the relations

$$\sqrt{\frac{J}{M}\frac{c_2}{c_1}} \le a_{\text{opt}} \le a_1,\; b_{\text{opt}} = \frac{c_1}{c_2} a_{\text{opt}} . \tag{12}$$

These provisions correspond to purely translational natural oscillations with a frequency equal to $\sqrt{\lambda_{\text{VT}}}$.

The case $c_2 b_1 < c_1 a_1$ is considered in a similar way, but now the permissible area is represented by another rectangle (Fig. 3 *a*). Omitting reasonings similar to those for the previous case, we present the results:

$$J \ge J_{ab} \Rightarrow a_{\text{opt}} = a_1,\ b_{\text{opt}} = b_1. \tag{13}$$

$$J_b \le J < J_{ab} \Rightarrow b_{\text{opt}} = b_1,\ a_{\text{opt}} = J/Mb_1,\ \lambda_{\max} = \frac{c_2}{M}\left(1+\frac{Mb_1^2}{J}\right), \text{ node on the left support.} \tag{14}$$

$$J < J_b \Rightarrow \sqrt{\frac{J}{M}\frac{c_1}{c_2}} \le b_{\text{opt}} \le b_1,\ a_{\text{opt}} = \frac{c_2}{c_1}b_{\text{opt}},\ \lambda_{\max} = \lambda_{\text{VT}} = \frac{c_1+c_2}{M}, \text{ the mode is horizontal.} \tag{15}$$

4.3.1.2. Problem 2. It is required to find the optimal positions of the supports under the given limitations

$$a \le a_1,\quad b \ge b_1. \tag{16}$$

Let us first consider the case $c_2 b_1 > c_1 a_1$, which corresponds to the permissible area in the form of a half-strip, bounded by bold straight lines (Fig. 3 *b*) above the straight line $b = (c_1/c_2)a$. The dash-dotted line shows the graph of dependence $b = J_{ab}/Ma$.

At $J \ge J_{ab}$, the boundary hyperbola $b = J/Ma$ (line 1 in Fig. 3 *b*) intersects the vertical border of the permissible area at a point P and, in accordance with the directions of the "arrows of growth" (Fig. 2), this point corresponds to the optimal position of the supports.

At $J < J_{ab}$ (line 2), the "arrows of growth" lead to the corner of the permissible area. So in the case $c_2 b_1 > c_1 a_1$

$$J \ge J_{ab} \Rightarrow a_{\text{opt}} = a_1,\ b_{\text{opt}} = \frac{J}{Ma_1},\ \lambda = \lambda_{\max} = \frac{c_1}{M}\left(1+\frac{Ma_1^2}{J}\right), \text{ node on the right support,} \tag{17}$$

$$J < J_{ab} \Rightarrow a_{\text{opt}} = a_1,\ b_{\text{opt}} = b_1. \tag{18}$$

In the case $c_2 b_1 < c_1 a_1$, the permissible area (bounded by thin lines in Fig. 3 *b*) is intersected by a straight line $b = (c_1/c_2)a$. The plane is divided by two dashed lines into three parts.

When $J > J_a$, the line $b = J/Ma$ (line 2) intersects the vertical border of the permissible area at a point Q that, as above, corresponds to the optimal position of the supports.

At $J_b < J \le J_a$ (line 3), a segment of the straight line $b = (c_1/c_2)a$ falls into the permissible area, the lower boundary of which depends on J. This segment corresponds to the optimal positions of the supports, at which the fundamental vibrations will be vertical and translational.

When $J \le J_b$, the entire part of the straight line $b = (c_1/c_2)a$ that belongs to the permissible band becomes available. Thus, for $c_2 b_1 < c_1 a_1$

$$J \ge J_a \Rightarrow a_{\text{opt}} = a_1,\ b_{\text{opt}} = \frac{J}{Ma_1},\ \lambda = \lambda_{\max} = \frac{c_1}{M}\left(1+\frac{Ma_1^2}{J}\right), \tag{19}$$

$$J_b < J \le J_a \Rightarrow \sqrt{\frac{J}{M}\frac{c_2}{c_1}} \le a_{\text{opt}} \le a_1,\ b_{\text{opt}} = \frac{c_1}{c_2}a_{\text{opt}},\ \lambda_{\max} = \lambda_{\text{VT}} = \frac{c_1+c_2}{M}. \tag{20}$$

$$J \le J_b \Rightarrow \sqrt{\frac{J_b}{M}\frac{c_2}{c_1}} = \frac{c_2}{c_1}b_1 \le a_{\text{opt}} \le a_1,\ b_{\text{opt}} = \frac{c_1}{c_2}a_{\text{opt}},\ \lambda_{\max} = \lambda_{\text{VT}} = \frac{c_1+c_2}{M}. \tag{21}$$

At $J \le J_a$, the optimal position of the supports corresponds to a horizontal mode.

https://doi.org/10.31650/2618-0650-2021-3-2-88-102

4.3.1.3. Problem 3. It is required to find the optimal positions of the supports under the given limitation

$$a + b = \ell = \text{const}. \tag{22}$$

The position of the supports, at which the natural vibrations are translational, is determined by the equality $c_1 a = c_2 b$, whence, taking into account (22)

$$a = \frac{c_2 \ell}{c_1 + c_2}, \; b = \frac{c_1 \ell}{c_1 + c_2}. \tag{23}$$

This position is optimal if at the same time $J \le abM$, i.e.

$$J \le M \frac{c_1 c_2}{(c_1 + c_2)^2} \ell^2. \tag{24}$$

Let us show that, under the condition (22), equality $c_1 a = c_2 b$ still provides a maximum for the fundamental frequency, even if it is realized through a rotational mode (of 2nd type), i.e. when the inequality opposite (24) is satisfied. To do this, recall (Sec. 4.2.6) that for $c_1 a = c_2 b$, the rotational mode has a node in the center of mass G $\left(X_1 = a,\, X_2 = b,\, r = 0\right)$ and it corresponds to the frequency determined by the equality

$$\lambda = \lambda_{\mathrm{R}} = \frac{\sum c_j X_j^2}{J + Mr^2} = \frac{c_1 a^2 + c_2 b^2}{J} = \frac{1}{J}\left[c_1\left(\frac{c_2 \ell}{c_1 + c_2}\right)^2 + c_2\left(\frac{c_1 \ell}{c_1 + c_2}\right)^2\right] = \frac{c_1 c_2 \ell^2}{(c_1 + c_2) J}, \tag{25}$$

moreover, $\lambda_{\mathrm{VT}} <> \lambda_{\mathrm{R}} \Leftrightarrow J <> abM$. Let's move the supports, breaking the equality $c_1 a = c_2 b$, but not changing $\ell = a + b = \text{const.}$. In this new position, we will install the rigid hinge at the same distances a and b from the supports. The formed system with one degree of freedom has a "rotational" frequency, the square of which

$$\lambda_{\mathrm{R}}^* = \frac{c_1 a^2 + c_2 b^2}{J + Mr^2} < \frac{c_1 a^2 + c_2 b^2}{J} = \lambda_{\mathrm{R}} \tag{26}$$

according to Sec. 4.2.1, is greater than the square of the fundamental frequency λ_1 of the system before the introduction of the hinge, from which it follows that as a result of the moving of the supports $\lambda_1 \le \lambda_{\mathrm{R}}^* < \lambda_{\mathrm{R}}$, i.e. λ_{R} is the maximum of λ_1.

Thus, the solution to the problem is determined by relations (23), and

$$\lambda_{\max} = \begin{cases} \lambda_{\mathrm{VT}} = \dfrac{c_1 + c_2}{M}, \text{ if } J \le M \dfrac{c_1 c_2}{(c_1 + c_2)^2} \ell^2, \\[2ex] \lambda_{\mathrm{R}} = \dfrac{c_1 c_2 \ell^2}{(c_1 + c_2) J} \le \lambda_{\mathrm{VT}}, \text{ if } J \ge M \dfrac{c_1 c_2}{(c_1 + c_2)^2} \ell^2. \end{cases} \tag{27}$$

4.3.1.4. Problem 4. It is required to find the optimal positions of the supports under the limitation expressing by inequality

$$a + b \le \ell = \text{const}. \tag{28}$$

If $J \ge M\left(c_1 c_2 / (c_1 + c_2)^2\right)\ell^2$, for any a and b, for which $a + b = \ell^* < \ell$, the inequality $J > M\left(c_1 c_2 / (c_1 + c_2)^2\right)\ell^{*2}$ holds, whence, according to (27), for these a and b $\lambda \le c_1 c_2 \ell^{*2} / (c_1 + c_2) J < c_1 c_2 \ell^2 / (c_1 + c_2) J$. Therefore, the maximum of frequency is reached when equality is satisfied in (28) and a and b are determined by relations (23). In this case $\lambda_{\max} = \lambda_{\mathrm{R}}$ (25).

If $J < Mc_1c_2\ell^2/(c_1+c_2)^2$, we define $\ell_J = \sqrt{J(c_1+c_2)^2/Mc_1c_2} < \ell$. For any $\ell \ge \ell_J$, there is the optimal position of the supports (23), described in Sec. 4.3.1.3, providing a translational fundamental mode. From here

$$\frac{c_2\ell_J}{c_1+c_2} \le a_{\mathrm{opt}} \le \frac{c_2\ell}{c_1+c_2},\ b_{\mathrm{opt}} = \frac{c_1}{c_2}a_{\mathrm{opt}},\ \lambda_{\max} = \lambda_{\mathrm{VT}} = \frac{c_1+c_2}{M}. \tag{29}$$

4.3.2. Multi-support model.

Problem 5. The values of M, J, x_j, c_j are specified, $j = 1, 2, \ldots, N$. The optimal position of the additional $(N+1)$-th support with a given stiffness coefficient c is sought.

Let us denote U – the original N-support model, U^* – the model formed from U by introducing an additional support, λ and λ^* – respectively, the squares of their frequencies.

The system (3) should be replaced with system

$$\left.\begin{aligned} &(C+c-M\lambda^*)Y + (S+cx)\Phi = 0 \\ &(S+cx)Y + \left(Q+cx^2 - J\lambda^*\right)\Phi = 0 \end{aligned}\right\}, \tag{30}$$

where x is a coordinate of the introduced support.

Based on Sec. 4.2.1, we conclude that for any position of this support

$$\lambda_1^* \le \lambda_{\mathrm{VT}}^* = \frac{C+c}{M}. \tag{31}$$

Let us determine $c_{\mathrm{cr}} = M\lambda_2 - C$ – the value of the stiffness coefficient of the introduced support, at which, in accordance with Sec. 4.2.2, translational natural vibrations of the EM U^* with a frequency $\sqrt{\lambda_2}$ are possible.

If $c < c_{\mathrm{cr}} = M\lambda_2 - C$, $\lambda_{\mathrm{VT}}^* < \lambda_2$ and the optimal coordinate of the movable support in accordance with Sec. 4.2.1 is equal to

$$x_{\mathrm{opt}} = -\frac{S}{c},\ \lambda_{1\max}^* = \lambda_{\mathrm{VT}}^* = \frac{C+c}{M}. \tag{32}$$

If $c \ge c_{\mathrm{cr}} = M\lambda_2 - C \Rightarrow \lambda_{\mathrm{VT}}^* \ge \lambda_2$, the value $\sqrt{\lambda_{\mathrm{VT}}^*}$ cannot be exceeded by the fundamental frequency of U^* after the imposition of one constraint by virtue of Rayleigh's theorem (Sec. 4.2.1), i.e. $\sqrt{\lambda_{\mathrm{VT}}^*} = \sqrt{\lambda_2^*}$ is the second frequency of the EM U^*. We'll show that in this case the optimal position of the support of the EM U^* is the node of the second mode of the EM U corresponding to the frequency $\sqrt{\lambda_2}$ (if it exists).

If we place an additional support of stiffness c in the node of the second mode of EM U, to which the natural frequency $\sqrt{\lambda_2}$ corresponds, this mode and the corresponding frequency, and, consequently, the distance r_2 from the node, do not depend on the value of c. Therefore, as relation (5) shows, also the value of r_1 and, consequently, the other mode of the model U^* do not depend on c. The corresponding frequency is determined according to (6) by the equality

$$\lambda_1^* = \frac{\sum c_j X_j^2 + cX^2}{J + Mr_1^2}, \qquad (33)$$

where X is the distance from the node of the 1st mode of U to the introduced support (to the node of the 2nd mode of U). From here it can be seen that with growth of c this frequency increases monotonically and indefinitely and at some moment becomes equal to $\sqrt{\lambda_2}$. Thus, the frequency $\sqrt{\lambda_2}$ becomes double, since it corresponds to two linearly independent modes, and, consequently, any linear combination of them, in particular, corresponding to vertical translational oscillations. Therefore $\lambda_{\text{VT}}^* = \lambda_2$, whence, taking into account (32), it follows that the corresponding value of c is equal to $c_{\text{cr}} = M\lambda_2 - C$.

With the growth of c above c_{cr}, the fundamental mode, without changing its configuration, becomes the second, because the corresponding frequency, determined from (33), becomes higher than $\sqrt{\lambda_2}$, which, thus, becomes the fundamental for U^*, $\lambda_1^* = \lambda_2$. By virtue of inequality (2), Sec. 4.2.1, $\lambda_1^* = \lambda_2 = \lambda_{1\max}^*$. Therefore at $c > c_{\text{cr}}$, the node of the second mode of the EM U is the optimal position of the additional support in the EM U^*. Its coordinate is

$$x_{\text{opt}} = -\frac{S}{c_{\text{cr}}} = -\frac{S}{M\lambda_2 - C}. \qquad (34)$$

The results (32), (34) make it possible to represent schematically the graph of the square of the fundamental frequency λ_1^* of the model U^* as a function of the position of the additional support.

If the second equation of system (30) is divided by x and x is tended to $\pm\infty$, we obtain the equalities

$$\lim_{x\to\pm\infty} \Phi x = -Y, \quad \lim_{x\to\pm\infty} \Phi = 0. \qquad (35)$$

From the first equation, taking into account (35), we come to the conclusion

$$x \to \pm\infty \Rightarrow (C - M\lambda^*)Y + cY + cx\Phi \to 0, \qquad (36)$$

whence

$$\lim_{x\to\pm\infty} \lambda^* = \frac{C}{M}. \qquad (37)$$

Based on the results obtained, you can sketch a schematic graph of dependence of λ_1^* on the coordinate x of the support (Fig. 4), which allows you to find its optimal position.

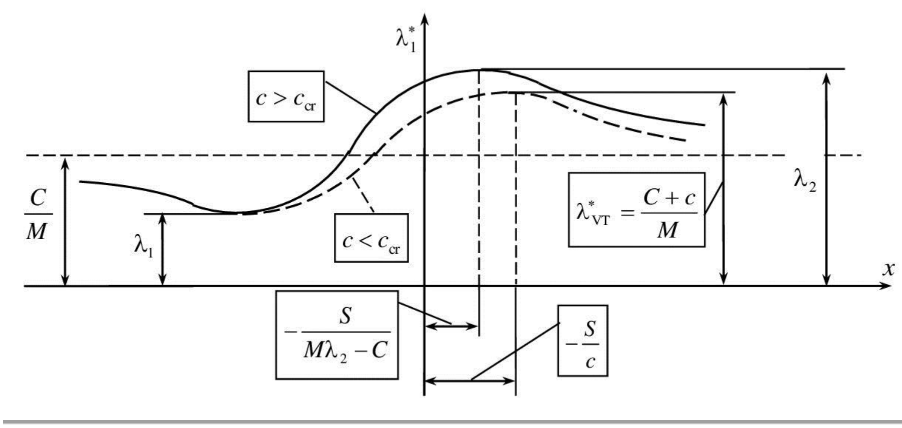

Fig. 4. Schematic graph of the dependence of the square of the fundamental frequency on the position x of the movable support

If the fundamental mode of EM U has a node, then it is to the left of the center of mass G. To verify this, consider the first of equations (3). In accordance with Rayleigh's theorem (Sec. 4.2.1) $\lambda_1 < \lambda_{VT} = C/M \Rightarrow C - M\lambda_1 > 0$. In combination with the condition $S < 0$ (Sec. 4.2), this means that the fundamental mode of EM U has Y and Φ of the same sign, i.e. that the node is to the left of the center of mass.

The function $\lambda_1^*(x)$ has two extrema: the node of the fundamental mode of EM U, to which the minimum $\lambda_{1\min}^* = \lambda_1$ corresponds, and a point with a coordinate x_{opt} determined from (32), if $c < c_{\text{cr}}$, and from (34), if $c \geq c_{\text{cr}}$ (the node of the second mode of EM U), to which the maximum $\lambda_{1\max}^* = \lambda_{VT}^*$, respectively $\lambda_{1\max}^* = \lambda_2$ corresponds. According to Sec. 4.2.3, the function $\lambda_1^*(x)$ has no other extrema, and these two extrema separate the areas of monotonicity. In fig. 4, the solid line corresponds to the case $c > c_{\text{cr}}$, and the dashed line – to the case $c < c_{\text{cr}}$. The results obtained make it possible to find the optimal positions of support under various constraints.

Example. Consider the problem of maximizing the fundamental frequency for the case $N = 2$, $x_1 = -a$, $x_2 = b$, $S = -(c_1 a - c_2 b)$. The optimal position of the third support is sought taking into account the limitations $-a \leq x \leq b$.

If $J \leq Mab$, the node of the second mode falls inside or on the right border of the allowed range $-a \leq x \leq b$, and the node of the first mode is outside it or on the left border.

If, in this case $c < (c_1 a - c_2 b)/b$, the maximum of the function $\lambda_1^*(x)$ according to (32) occurs at $x > b$, and $\lambda_1^*(x)$ monotonically increases in the admissible range, whence

$$x_{\text{opt}} = b,\ \lambda_{1\max}^* = \lambda_1^*(b).$$

If $(c_1 a - c_2 b)/b \leq c < c_{\text{cr}} = M\lambda_2 - (c_1 + c_2)$, according to (32) $x_{\text{opt}} = (c_1 a - c_2 b)/c \leq b$, $\lambda_{1\max}^* = \lambda_{VT}^* = (c_1 + c_2 + c)/M$.

Finally, if $c \geq c_{\text{cr}} = M\lambda_2 - (c_1 + c_2)$, the optimal position is a node of the second mode and $\lambda_{1\max}^* = \lambda_2$.

In the case $J > Mab$, only the node of the first mode, which corresponds to the minimum of $\lambda_1^*(x)$, falls inside the permissible range $-a \leq x \leq b$, and when the support moves to both sides from this node, $\lambda_1^*(x)$ monotonically increases within this range. It follows from this

$$x_{\text{opt}} = \begin{cases} -a, \text{ if } \lambda_1^*(-a) > \lambda_1^*(b), \\ b, \text{ if } \lambda_1^*(-a) < \lambda_1^*(b), \\ -a \text{ and } b, \text{ if } \lambda_1^*(-a) = \lambda_1^*(b). \end{cases}$$

And in this case $\lambda_{1\max}^* = \max\{\lambda_1^*(-a), \lambda_1^*(b)\}$.

5 RESEARCH RESULTS DISCUSSION

The performed research made it possible to find the position of elastic supports, which provides the maximum of the fundamental natural frequency of an elastically fixed rigid body. In particular, the conditions are described under which the maximum of frequency is achieved with purely translational vibrations of the body. These results greatly facilitate the determination of these positions and do not require cumbersome calculations using a formal mathematical optimization technique. They also make it possible to qualitatively trace the behavior of the natural frequency with some changes in the parameters of the model. The approach used allows its optimization taking into account various limitations..

6 CONCLUSIONS

A practically convenient approach for determining the position of elastic supports, which maximally increases the natural frequency of an elastically supported body, is proposed. It makes it possible to qualitatively characterize this position depending on the given design parameters. The results obtained can be used in the design and operation of various engineering structures in order to create conditions that ensure their reliable operation. They allow us to outline the directions for further research, including in consideration additional degrees of freedom of the body and a wider variety of elastic connections.

References

1. Rozvany, G. I. N., Lewiński, T. (Eds) (2014). *Topology Optimization in Structural and Continuum Mechanics*. CISM International Centre for Mechanical Sciences. Springer. [in English].
2. Reytman, M. I., Shapiro, G. S. (1976). *Metody optimal'nogo proektirovanyia deformiruemykh tel*. [Methods of Optimal Design of Deformable Bodies]. M.: Nauka. [in Russian].
3. Spiekermann, C. E., Radcliffe, C. J., Goodman, E. D. (1985). Optimal Design and Simulation of Vibrational Isolation Systems. *Journal of Mechanisms, Transmissions, and Automation in Design*. Vol. 107. Trans. ASME. 271–276.
4. Strutt, J. W. (Reyleigh). (1877). *Theory of Sound*, vol. I. Macmillan and Co. London. [in English].
5. Bekshaev, S. Ya., Koshkin, L. V., Nudel'man, Ya. L. (1976). K voprosu ob optimal'nom raspolozhenii mass i opor vibriruyucshego sterzhnya [On the question of the optimal arrangement of masses and supports of a vibrating rod]. *Sudostroenie I sudoremont* [Shipbuilding and ship repair]. M.: Reklaminformburo MMF. 7. 64 – 67. [in Russian].
6. Bekshaev, S, Ya. (2007). O vozmuscheniyakh sobstvennykh chastot vantovo-sterzhnevykh sistem, podkreplennykh sviaziami [On perturbations of eigenfrequencies of cable-stayed rod systems, reinforced by constraints]. *Visnyk Odes'koi derzhavnoi akademii budivnitstva ta arkhitektury* [Bulletin of the Odessa State Academy of Education and Architecture]. 26. 40 –53. [in Russian].
7. Bekshaev, S. (2020). Yakisni vlastyvosti kolyvan' pruzhno obpertogo tverdogo tila [Qualitative properties of vibrations of elastically supported rigid body]. *Mechanics and Mathematical Methods*. 2(2). 85–94.
8. Loytsyansky, L. G., Lur'e, A. I. (1934). *Teoreticheskaya mekhanika* [Theoretical mechanics]. ch. III. M.-L.: GTTI. [in Russian].
9. Meirovitch, L. (2001). *Fundamentals of Vibrations*. McGraw-Hill. [in English].
10. Inman, Daniel J. (2014) *Engineering Vibrations*. 4-th ed. Pearson. [in English].

Література

1. Rozvany G. I. N., Lewiński T. (eds). Topology Optimization in Structural and Continuum Mechanics. CISM International Centre for Mechanical Sciences. Springer. 2014. 471 p.
2. Рейтман М. И., Шапиро Г.С. Методы оптимального проектирования деформируемых тел. М.: Наука, 1976. 267 с.
3. Spiekermann C. E., Radcliffe C. J., Goodman E.D. Optimal Design and Simulation of Vibrational Isolation Systems. Journal of Mechanisms, Transmissions, and Automation in Design. Vol. 107. Trans. ASME. 1985. pp. 271–276.
4. Strutt, J. W. (Reyleigh). Theory of Sound, vol. I. Macmillan and Co. London. 1877. 326 + XI p.
5. Бекшаев С. Я., Кошкин Л. В., Нудельман Я. Л. К вопросу об оптимальном расположении масс и опор вибрирующего стержня. Судостроение и судоремонт. М.: Рекламинформбюро ММФ, 1976. Вып. VII. С. 64–67.

6. Бекшаев С.Я. О возмущениях собственных частот вантово-стержневых систем, подкрепленных связями. Вісник Одеської державної академії будівництва та архітектури, 2007. №26. С. 40–53.
7. Бекшаєв С. Я. Якісні властивості коливань пружно обпертого твердого тіла. Механіка та математичні методи, 2020. том 2. вип. 2. С. 85–94.
8. Лойцянский Л. Г., Лурье А. И. Теоретическая механика, ч. III. М.-Л.: ГТТИ. 1934. 624 с.
9. Meirovitch L. Fundamentals of Vibrations. McGraw-Hill, 2001. 806 + XVIII p.
10. Inman Daniel J. Engineering Vibrations. 4-th ed. Pearson. 2014. 707 + XII p.

Sergey Bekshaev
Odessa State Academy of Civil Engineering and Architecture
Senior Lecturer
Didrihsona str., 4, Odessa, Ukraine 65029
s.bekshayev@gmail.com
ORCID: 0000-0002-5752-5321

For references:
Bekshaev S. (2021). Some problems of optimization and control of the natural frequencies of an elastically supported rigid body. Mechanics and Mathematical Methods. 3 (2). 88–102.

Для посилань:
Бекшаєв С. Я. Деякі задачі оптимізації і управління власними частотами пружно опертого твердого тіла. Механіка та математичні методи, 2021. Т. 3. № 2. С. 88–102.