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THE PROBLEM OF SHOCK WAVE DIFFRACTION BY A RIGID HEMISPHERICAL INCLUSION

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Abstract: Currently, a large number of various components, parts, and entire assemblies that make up modern technical equipment – and even entire structures – are forced to operate either in continuous (stationary) or non-stationary modes due to rapid changes in external forces and loads that vary over time. Thin-walled shell-type structures constitute a very broad class of mechanical equipment found in modern surface, underwater, and air transportation, in rocket technology, and in construction. Under intense loads of various origins during the operation of such structures and mechanisms – including those of an explosive nature – high stresses arise, which are highly undesirable from the standpoint of their strength and reliability. Among the general range of problems associated with the use of shell structures and designs, including thin-walled ones, a class of problems should be highlighted that involves studying the interaction of undeformed shells of various geometries with the surrounding elastic medium.

This paper presents an approach to studying steady-state (transient) wave diffraction problems at a spherical defect, based on the application of the method of discontinuous solutions, which is founded on the method of generalized integral transforms [1,2]. A defect is defined as a portion of a surface such that, when crossed, wave functions, displacements, and stresses undergo first-kind discontinuities. Another type of defect is a thin rigid shell, the mid-surface of which coincides with the same portion of the surface. For this case, a discontinuous solution to the wave equation of linear elasticity theory in a spherical coordinate system has been derived. The problem of diffraction of a plane shock wave at an arbitrary angle of incidence α onto a rigid, thin-walled, stationary hemispherical object coupled to a bottom structure has been solved. The Heaviside unit function was chosen as the function describing such a wave. It is convenient to use in this case because the wave process formed in the elastic medium so that the wave process formed in the elastic medium acts step-wise (discretely) starting from the time moment t corresponding to the arrival of the wave-front moving at a prescribed speed C .

Keywords: wave equation, elasticity theory, discontinuous solution, jump, shock-wave diffraction, spherical coordinates, spherical defect.

ЗАДАЧА ДИФРАКЦІЇ УДАРНОЇ ХВИЛІ НА ЖОРСТКОМУ НАПІВСФЕРИЧНОМУ ВКЛЮЧЕННІ

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Анотація: В даний час велика кількість різноманітних вузлів, деталей та цілих агрегатів, які входять у склад сучасних технічних обладнань та, навіть, цілих споруд вимушені працювати або у постійно діючих (стаціонарних), або у нестаціонарних режимах внаслідок швидких змін зовнішніх сил та навантажень, що впливають у часі. Тонкостінні конструкції оболонкового типу складають дуже широкий клас механічних обладнань, що зустрічаються у сучасному надводному, підводному та повітряному транспорті, у ракетній техніці, а також у будівництві. При інтенсивних навантаженнях різного походження в умовах експлуатації таких конструкцій та механізмів, в тому числі і вибухового типу, виникають великі напруження, які зовсім небажані з точки зору їх міцності та надійності. З загального кола проблем, що пов'язані з використанням оболонкових структур і конструкцій, в тому числі і тонкостінних, слід виділити



клас задач, що пов'язаний з вивченням взаємодії недеформованих оболонок різної геометрії з навколишнім пружним середовищем.

У запропонованій роботі розроблено підхід до дослідження стаціонарних (нестаціонарних) задач дифракції хвиль на сферичному дефекті, заснований на застосуванні методу розривних рішень, який базується на методі узагальнених інтегральних перетворень [1,2]. Під дефектом слід розуміти таку частину поверхні, при перетинанні якої, зазнають розривів першого роду хвильові функції, зміщення та напруження. Іншим випадком дефекту є тонка жорстка оболонка, серединна поверхня якої співпадає з тією ж частиною поверхні. Для цього випадку побудовано розривний розв'язок хвильового рівняння лінійної теорії пружності в сферичній системі координат. Розв'язано задачу дифракції ударної плоскої хвилі при довільному куті фронту атаки α на жорсткий тонкостінний нерухомий напівсферичний об'єкт, зчеплений з донним утворенням. У якості функції, що описує таку хвилю, було обрано одиничну функцію Хевісайда. Її зручно використовувати в цьому випадку тому, щоб хвильовий процес, який формується в пружному середовищі, працював дискретно (східчасто) тільки в той момент часу t , коли буде надходити черговий фронт зі заздалегідь заданою швидкістю C .

Ключові слова: хвильове рівняння, теорія пружності, розривний розв'язок, стрибок, стаціонарна дифракція, сферичні координати, дефект.

1 INTRODUCTION

The importance and practical relevance of studying the interaction between undeformable shells of various geometries and the surrounding elastic medium are associated with the need to improve the resistance of structures, such as vessels, to underwater and airborne explosions, to develop and refine methods for underwater and atmospheric acoustics, and to ensure the seismic stability of certain aerospace and hydraulic engineering structures and their components. Thus, the development of diverse mathematical methods for solving problems related to the interaction of non-stationary (stationary) waves of arbitrary nature with various objects, including shell-type structures, remains a highly relevant topic today, which ultimately defined the scope of this paper.

2 LITERATURE REVIEW AND PROBLEM STATEMENT

Significant contributions to the development of theory and methods for solving the dynamic problems of bodies interacting with the surrounding medium have been made by scientists such as E. P. Babailov, V. A. Babeshko, A. I. Babichev, O. T. Buriev, M. D. Veksler, O. M. Guz, O. D. Goldenveizer, O. M. Kovshov, V. D. Kubenko, I. T. Seleznev, O. V. Tarlakovsky, G. Ya. Popov, and many others.

The methods developed to date for solving three-dimensional boundary-value problems in continuum mechanics can be broadly categorized into analytical and numerical approaches. Among the analytical methods, the following key approaches stand out: integral equation methods (potential theory); the method of separation of variables and its various modifications (the Fourier method and its generalizations in vector and scalar forms, as well as reduction to infinite systems of algebraic equations); the method of functions of a complex variable [3–7].

In addition, to solve complex boundary value problems involving the nonstationary scattering of elastic waves on a spherical inclusion, authors such as B. Sh. Usmonov and R. T. Mukhitdinov, representing the Tashkent Chemical-Technological Institute, employ specific hybrid (combined) analytical methods. These methodological approaches are primarily based on the application of Fourier integral transforms in the time domain. However, a significant mathematical difficulty arises because the eigenfunctions of the boundary value problem under consideration cannot be rigorously treated as ordinary vectors in a classical Hilbert space, since they fail to satisfy the required normalization conditions due to their exponential growth with increasing distance. Therefore, it becomes necessary to additionally employ generalized functions together with specialized mathematical methods developed within the framework of modern scattering theory.

Nevertheless, these methods are mainly applicable to canonical domains where boundary surface equations can be reduced to standard canonical forms. For spherical objects with non-standard geometry (such as a sphere with a circular aperture or a spherical segment in the presence of neighboring objects, e.g., a cone and an additionally segmented sphere), the principal part of the matrix operator for the plane acoustic wave diffraction problem is formulated. Problem regularization relies on an Abel-type integral transform, which allows reducing the system to an infinite set of Fredholm algebraic equations of the second kind with a compact operator in a Hilbert space [8–9].

Currently, various numerical approaches – such as finite-difference and finite-element methods – have gained widespread use in solving three-dimensional problems. The present work is devoted to solving a spatial problem in the theory of elasticity for a hemispherical segment bonded to a rigid base using the discontinuous solution method [1, 2].

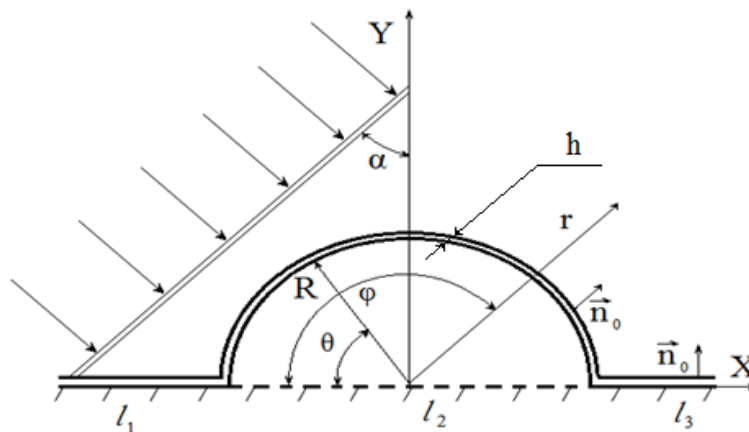
3 RESEARCH AIM AND OBJECTIVES

The objective of this study is to extend the discontinuous solution method [1, 2] to dynamic problems involving the diffraction of shock waves by spherical defects. To achieve this, the following tasks are addressed: construct a universal discontinuous solution to the wave equation and use it to reduce the problem of shock wave diffraction [3] on a thin, rigid hemispherical inclusion bonded to a rigid base to an integro-differential equation. Conduct an analytical convergence analysis and examine the asymptotic expressions for the resulting equation's solution. Compare the obtained solution with the findings of [11, 12, 13] regarding the reactive torsional moment of a spherical defect in an elastic medium induced by a torsional wave [5], expressed through displacements and stress components [3, 10, 11] dependent on wave functions [14, 15] which, according to the proposed approach, must satisfy the discontinuous solutions of the wave equations.

4 RESEARCH RESULTS

4.1 Problem Statement

Suppose that an infinite elastic medium contains a rigid hemispherical inclusion in the shape of a hemispherical segment of thickness h , which is rigidly bonded to a bottom surface (base). Furthermore, assume that this inhomogeneity is rigidly bonded to the surrounding space and remains "frozen" (stationary) within it. Let a plane shock wave impinge on the half-space, with the geometric parameters of the system being such that superimposition of the waves reflected and incident from the sphere and the bottom occurs. It is required to determine the wave superposition potential using the discontinuous solution method, which must satisfy the boundary conditions everywhere except at the points of the defect itself.



Thus, in dimensionless variables, the problem reduces to finding a solution of the wave equation in spherical coordinates [16]:

$$\square \Psi = \Delta \Psi - \frac{1}{a^2} \frac{\partial^2 \Psi}{\partial t^2} = 0, \quad (1)$$

under the initial conditions:

$$\Psi|_{t=0} = 0, \quad \frac{\partial \Psi}{\partial t} \Big|_{t=0} = 0,$$

and boundary conditions on the surface $l = l_1 + l_2 + l_3$:

$$\left. \frac{\partial \Psi}{\partial \varphi} \right|_{\substack{r > R, \\ \varphi = -\pi, \pi}} = 0, \quad \left. \frac{\partial \Psi}{\partial r} \right|_{\substack{r = R, \\ \varphi = -\pi, \pi}} = 0,$$

where $\square$ denotes the d'Alembertian operator (Hamilton operator), Δ is the Laplace operator, and r, θ, φ are the radial and angular spherical coordinates, with $0 < \theta < \pi$.

It is also assumed that the wave front forms an angle α with the OY -axis, where R denotes the radius of the spherical inclusion.

Assuming that the total wave potential can be decomposed into three components [14]:

Ψ_{inc} – the incident-wave potential;

Ψ_{refl} – the reflected-wave potential;

Ψ_{diff} – the diffracted-wave potential,

the boundary conditions take the expanded form:

$$\left. \frac{\partial \Psi_{diff}}{\partial r} \right|_{\substack{r = R, \\ \varphi = -\pi, \pi}} = - \left(\left. \frac{\partial \Psi_{refl}}{\partial r} + \frac{\partial \Psi_{inc}}{\partial r} \right) \right|_{\substack{r = R, \\ \varphi = -\pi, \pi}}. \quad (2)$$

Furthermore, the case of a rigid shell yields the following relationship:

$$\left(\left. \frac{\partial \Psi_{inc}}{\partial \varphi} + \frac{\partial \Psi_{refl}}{\partial \varphi} \right) \right|_{\substack{r > R, \\ \varphi = -\pi, \pi}} = 0. \quad (3)$$

4.2 Construction of a Discontinuous Solution in Spherical Coordinates

Expanding the wave equation (1) in spherical coordinates, we obtain (see [16]):

$$\square \Psi(r, \theta, \varphi, t) = \Delta \Psi(r, \theta, \varphi, t) - \frac{1}{a^2} \frac{\partial^2 \Psi(r, \theta, \varphi, t)}{\partial t^2} = 0, \quad (4)$$

where:

$$\Delta \Psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \varphi^2}. \quad (5)$$

In accordance with [1, 2], the jumps in the function and its normal derivative across the surface are denoted as follows:

$$\langle \Psi(R, \theta, \varphi, t) \rangle = \Psi(R-0, \theta, \varphi, t) - \Psi(R+0, \theta, \varphi, t), \quad (6)$$

$$\langle \Psi'(R, \theta, \varphi, t) \rangle = \Psi'(R-0, \theta, \varphi, t) - \Psi'(R+0, \theta, \varphi, t). \quad (7)$$

4.2.1 Elimination of the Variable t

We apply the Laplace transform [7] with respect to the time variable t (which plays the role of time in this context) to each of the expressions (4) – (7):

$$\Psi_p(r, \theta, \varphi) = \int_0^\infty \Psi(r, \theta, \varphi, t) \cdot \exp(-pt) dt. \quad (8)$$

Furthermore, to solve the problem in dimensionless variables, we additionally introduce a variable $\tau = at$ over which the integration is performed.

Thus, we obtain the following equation in terms of the Laplace transform:

$$\Delta \Psi_p(r, \theta, \varphi) - p^2 \Psi_p(r, \theta, \varphi) = 0,$$

where

$$\begin{aligned} \Delta \Psi_p(r, \theta, \varphi) = & \frac{2}{r} \frac{\partial}{\partial r} \Psi_p(r, \theta, \varphi) + \frac{\partial^2}{\partial r^2} \Psi_p(r, \theta, \varphi) + \\ & + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial \Psi_p(r, \theta, \varphi)}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \Psi_p(r, \theta, \varphi) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \Psi_p(r, \theta, \varphi). \end{aligned} \quad (9)$$

In turn, the jumps in the transforms and their derivatives (6), (7) are written as follows:

$$\langle \Psi_p(R, \theta, \varphi) \rangle = \Psi_p(R-0, \theta, \varphi) - \Psi_p(R+0, \theta, \varphi), \quad (10)$$

$$\langle \Psi'_p(R, \theta, \varphi) \rangle = \Psi'_p(R-0, \theta, \varphi) - \Psi'_p(R+0, \theta, \varphi). \quad (11)$$

4.2.2 Elimination of the Angle φ

We employ the Fourier integral transform [7]:

$$\Psi_{pn}(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi_p(r, \theta, \varphi) \exp(-in\varphi) d\varphi. \quad (12)$$

Applying this transform to all formulas (9) – (11), we yield the following expressions in terms of Fourier transforms:

$$\Delta \Psi_{pn}(r, \theta) - p^2 \Psi_{pn}(r, \theta) = 0,$$

where

$$\Delta \Psi_{pn}(r, \theta) = \frac{(r^2 \Psi'_{pn}(r, \theta))'}{r^2} + \frac{(\Psi'_{pn}(r, \theta) \sin \theta)'}{r^2 \sin \theta} - \frac{n^2}{r^2 \sin^2 \theta} \Psi_{pn}(r, \theta), \quad (13)$$

$$\langle \Psi_{pn}(R, \theta) \rangle = \Psi_{pn}(R-0, \theta) - \Psi_{pn}(R+0, \theta), \quad (14)$$

$$\langle \Psi'_{pn}(R, \theta) \rangle = \Psi'_{pn}(R-0, \theta) - \Psi'_{pn}(R+0, \theta). \quad (15)$$

In formula (11), (13) and below, we adopt the following notation: $(*)'$ – derivative with respect to the variable r and $(*)^\bullet$ – derivative with respect to the variable θ .

4.2.3 Elimination of the Angle θ

To eliminate θ , we use the Legendre integral transform [7]:

$$\Psi_{pnk}(r) = \int_0^\pi \sin \theta \cdot \Psi_{pn}(r, \theta) \cdot P_k^{[n]}(\cos \theta) d\theta, \quad (16)$$

where $P_k^{[n]}(\cos \theta)$ is the associated Legendre function, which arises when solving the Laplace equation in spherical coordinates.

Thus, applying the Legendre transform to equations (14), (15) yields equations depending solely on the radial variable r :

$$\Delta \Psi_{pnk}(r) - p^2 \Psi_{pnk}(r) = 0,$$

where

$$\Psi_{pnk}(r) = \frac{2}{r} \frac{d}{dr} \Psi_{pnk}(r) + \frac{d^2}{dr^2} \Psi_{pnk}(r) - \frac{k(k+1)}{r^2} \Psi_{pnk}(r). \quad (17)$$

In turn, the expressions for the jumps are written as:

$$\langle \Psi_{pnk}(R) \rangle = \Psi_{pnk}(R-0) - \Psi_{pnk}(R+0), \quad (18)$$

$$\langle \Psi'_{pnk}(R) \rangle = \Psi'_{pnk}(R-0) - \Psi'_{pnk}(R+0). \quad (19)$$

4.2.4 Application of the Hankel Transform with Respect to the Variable r

Combining equations (14), (15) into a single equation, as well as introducing a new function (for convenience of integration):

$$\Psi_{pnk}(r) = \frac{\Phi_{pnk}(r)}{\sqrt{r}}, \quad (20)$$

we obtain the following equation with respect to the function $\Phi_{pnk}(r)$:

$$\frac{1}{r} (r \Phi'_{pnk}(r))' - \frac{\Phi_{pnk}(r)}{r^2} \left(k + \frac{1}{2} \right)^2 - p^2 \Phi_{pnk}(r) = 0. \quad (21)$$

Let us apply the integral Hankel transform [7] to equation (21), which has the following form:

$$\Phi_{pnk\alpha} = \int_0^\infty r J_\nu(\alpha r) \Phi_{pnk}(r) dr, \quad (22)$$

where $J_\nu(\alpha r)$ is a cylindrical Bessel function of the first kind, α is the Hankel transform parameter, and $\nu = k + \frac{1}{2}$.

Integrating each term of (20) using formula (22), we obtain functions of the form $\Phi_{pnk\alpha}$ in terms of the Hankel transform; that is, we eliminate the variable r . At this stage of the solution, additional attention should be paid to the integrated (non-integral) terms with respect to the variable r on the interval from 0 to infinity. According to the discontinuous solution method [1,2], upon crossing the boundary points of the surface of a spherical defect of radius

R , (i.e., when $r \rightarrow R \pm 0$), the functions and their normal derivatives exhibit jump discontinuities (discontinuities of the first kind). Consequently, jumps

$$\begin{aligned}\langle \Phi_{pnk}(R) \rangle &= \Phi_{pnk}(R-0) - \Phi_{pnk}(R+0), \\ \langle \Phi'_{pnk}(R) \rangle &= \Phi'_{pnk}(R-0) - \Phi'_{pnk}(R+0),\end{aligned}$$

are formed, which need to be determined using the boundary conditions.

At this stage of the solution, the unknown function $\Phi_{pnk\alpha}$ will be expressed through the jumps as follows:

$$\Phi_{pnk\alpha} = \frac{R \left[\langle \Phi'_{pnk}(R) \rangle J_\nu(\alpha r) - \langle \Phi_{pnk}(R) \rangle \frac{\partial}{\partial R} J_\nu(\alpha r) \right]}{\alpha^2 + q^2}, \quad (23)$$

where $q = p$.

4.2.5 Application of the Inverse Hankel Transform

In order to restore r in the transform $\Phi_{pnk\alpha}$, one should use the inverse Hankel transform [7] with respect to the variable α :

$$\Phi_{pnk}(r) = \int_0^\infty \alpha J_\nu(\alpha r) \Phi_{pnk\alpha} d\alpha. \quad (24)$$

Applying (24) to both sides of equation (25), and taking into account that the jumps, $\langle \Phi'_{pnk}(R) \rangle$ and $\langle \Phi_{pnk}(R) \rangle$, according to the introduced definitions, are certain numerical expressions, and additionally using the integral transform formula from [17]:

$$\int_0^\infty \frac{x J_\nu(ax) J_\nu(bx)}{x^2 + c^2} dx = \begin{cases} I_\nu(bc) K_\nu(ac), & 0 < b < a, \\ I_\nu(ac) K_\nu(bc), & 0 < a < b, \end{cases} \quad \text{Re}(c) > 0, \text{Re}(\nu) > -1,$$

we obtain the following equation instead of (23):

$$\Phi_{pnk}(r) = R \left[\langle \Phi'_{pnk}(R) \rangle \overline{\Psi}_k(r, R) - \langle \Phi_{pnk}(R) \rangle \frac{\partial}{\partial R} \overline{\Psi}_k(r, R) \right], \quad (25)$$

where $I_\nu(bc)$, $K_\nu(ac)$ are the modified Bessel and Macdonald functions [19], respectively, $\nu = k + \frac{1}{2}$, $a = r$, $b = R$, $c = q$ and, in turn:

$$\overline{\Psi}_k(r, R) = \begin{cases} I_\nu(Rq) K_\nu(rq), & R < r; \\ I_\nu(rq) K_\nu(Rq), & r < R, \end{cases} \quad q = \omega \sqrt{\frac{\rho_m}{\mu}}, \quad (26)$$

q is the wavenumber, ω is the angular frequency, ρ_m is the material density, and μ is the Lamé constant.

To obtain the solution to the one-dimensional boundary value problem (17), it is necessary to again use relation (20), and then the discontinuous solution is finally written in the form:

$$\Psi_{pnk}(r) = R \left[\langle \Psi'_{pnk}(R) \rangle F_k(r, R) - \langle \Psi_{pnk}(R) \rangle \frac{\partial}{\partial R} F_k(r, R) \right], \quad (27)$$

where $F_k(r, R) = \frac{\overline{\Psi}_k(r, R)}{\sqrt{rR}}$.

Equation (27) contains two unknown jumps: $\Psi'_{pnk}(R)$ and $\Psi_{pnk}(R)$, the values of which must be determined. For this purpose, we first find $\frac{\partial}{\partial R} F_k(r, R)$. Further, taking the limit in (27) as the variable $r \rightarrow R+0$ (the case of a rigid inclusion), and also taking into account that, according to (18), (19):

$$\langle \Psi'_{pnk}(R) \rangle = -\Psi'_{pnk}(R+0), \quad \langle \Psi_{pnk}(R) \rangle = -\Psi_{pnk}(R+0) \quad (28)$$

respectively, and additionally introducing the notation $\Psi'_{pnk}(R+0) = f_{pnk}$, we obtain an equation for determining the unknown jump:

$$\Psi_{pnk}(R+0) = f_{pnk} \frac{2\nu K_\nu(Rq) - 2RqK_{\nu+1}(Rq)}{(2\nu+1)K_\nu(Rq) - 2RqK_{\nu+1}(Rq)}. \quad (29)$$

In turn, the function

$$f_{pnk} = \int_{-\pi}^{\pi} \int_0^{\pi} f_p(r, \theta, \varphi) \sin \theta P_k^{[n]}(\cos \theta) \exp(-in\varphi) d\varphi d\theta$$

is known, and its explicit form depends exclusively on the type of the chosen incident wave.

Using the expression for $F_k(r, R)$ from (27), formulas (26) and (29), the assumptions for the jumps (28), as well as transformations for the functions $I_\nu(Rq)K_\nu(rq)$ [19], we obtain the expression for the solution to equation (27) in the form:

$$\Psi_{pnk}(r) = \frac{2R^2 K_\nu(rq) f_{pnk}}{\sqrt{rR} [(2\nu-1)K_\nu(Rq) - 2RqK_{\nu+1}(Rq)]}. \quad (30)$$

4.2.6 Application of the Inverse Legendre Transform

To restore the angle θ in the transform $\Psi_{pnk}(r)$, one should use the inverse Legendre transform [7], which is written in the form:

$$\Psi_{pn}(r, \theta) = \sum_{k=|n|}^{\infty} \sigma_{kn} \Psi_{pnk}(r) P_k^{[n]}(\cos \theta), \quad (31)$$

where $\sigma_{kn} = \frac{(k-|n|)!}{(k+|n|)!} \left(k + \frac{1}{2} \right)$.

Applying series (31) to (30) and formally replacing the angle θ with ϑ in the forward Legendre transform (16), we obtain the following equation:

$$\Psi_{pn}(r, \theta) = 2R \sqrt{\frac{R}{r}} \int_0^{\pi} f_{pn}(\vartheta) \sin \vartheta \Omega_n(R+0, \theta, \vartheta) d\vartheta, \quad (32)$$

$$\text{where } \Omega_n(R+0, \theta, \vartheta) = \sum_{k=|n|}^{\infty} \sigma_{kn} P_k^{(|n|)}(\cos \theta) P_k^{(|n|)}(\cos \vartheta) \frac{K_\nu(rq)}{\left[(2\nu-1)K_\nu(Rq) - 2RqK_{\nu+1}(Rq) \right]}.$$

4.2.7 Application of the Inverse Fourier Transform

To restore the angle φ in the transform $\Psi_{pn}(r, \theta)$, one should use the inverse Fourier transform [7], which is written as:

$$\Psi_p(r, \theta, \varphi) = \sum_{n=-\infty}^{\infty} \Psi_{pn}(r, \theta) \cdot \exp(in\varphi). \quad (33)$$

Applying series (33) to equation (32), we formally replace the angle φ with ψ in the forward Fourier transform (12). Furthermore, by even extension of the functions on the right-hand side of (32), only the cosine term remains from the exponential, yielding the following equation:

$$\Psi_p(r, \theta, \varphi) = \frac{R}{\pi} \sqrt{\frac{R}{r}} \int_0^\pi \int_0^\pi f_p(\psi, \vartheta) \sin(\vartheta) \Omega(R+0, \theta, \vartheta, \varphi, \psi) d\psi d\vartheta, \quad (34)$$

where

$$\Omega(R+0, \theta, \vartheta, \varphi, \psi) = \sum_{n=-\infty}^{\infty} \sum_{k=|n|}^{\infty} \sigma_{kn} P_k^{(|n|)}(\cos \theta) P_k^{(|n|)}(\cos \vartheta) \cos(n\psi) \cos(n\varphi) \Lambda_\nu(r, R, q)$$

and

$$\Lambda_\nu(r, R, q) = \frac{K_\nu(rq)}{\left[(2\nu-1)K_\nu(Rq) - 2RqK_{\nu+1}(Rq) \right]}. \quad (35)$$

To proceed with solving the problem at hand, it is necessary at this stage to clarify the behavior of the sums and integrals involved in formulas (34) and (35).

4.3 Proof of Solution Convergence

The evaluation of definite integrals involving combinations of associated Legendre functions as terms of nested double series depends primarily on the indices k and $|n|$. Based on the orthogonality relations [19], the result of the integral evaluation branches into two cases. If the expression $(k+|n|)$ is an odd number, the terms of the series vanish (equal zero). If this expression is an even number, the result can be written in explicit analytical form [19] as a ratio of Gamma functions. Numerical test evaluations showed that as the lower index k (which always starts from $|n|$) increases, the terms of the series decay faster, thus ensuring the convergence of the series.

In addition, the behavior of the kernel $\Lambda_\nu(r, R, q)$ from formula (35) – represented by a combination of Macdonald functions (modified Bessel functions of the second kind) – at infinity and at zero for fixed and arbitrary values of the index k is of particular interest.

Thus, as the index k as the index tends to infinity while the argument of the function remains fixed [19], we have:

$$\frac{K_\nu(z)}{\left[(2\nu-1)K_\nu(z)-2zK_{\nu+1}(z)\right]} \sim \quad (36)$$

$$\sim \frac{\frac{1}{2}\Gamma(\nu)(z/2)^{-\nu}}{\left[(\nu-1/2)\Gamma(\nu)(z/2)^{-\nu}-2z\Gamma(\nu+1)(z/2)^{-\nu-1}\right]} = -\frac{1}{2k+1}, \text{ when } k \rightarrow \infty,$$

where $\Gamma(\nu)$ is the Gamma function, and $\nu = k + \frac{1}{2}$.

Let us consider the asymptotic expansion for large values of z and a fixed index k , having previously denoted $\mu = 4\nu^2$

$$\frac{K_\nu(z)}{\left[(2\nu-1)K_\nu(z)-2zK_{\nu+1}(z)\right]} \sim \quad (37)$$

$$\sim \frac{\sqrt{\frac{\pi}{2z}} \cdot \exp(-z) \left\{1 + \frac{\mu-1}{8z} + \dots\right\}}{\left((2\nu-1)-2z\right) \sqrt{\frac{\pi}{2z}} \cdot \exp(-z) \left\{1 + \frac{\mu-1}{8z} + \dots\right\}} = -\frac{1}{2(k-z)}, \text{ when } z \rightarrow \infty$$

Consequently, both series (36) – (37) converge under the respective conditions, which guarantees the convergence of the kernel containing the combinations of Macdonald functions (33).

4.4 Construction of the Incident Wave Potential

It is known that the solutions to the three-dimensional wave equation in a Cartesian coordinate system:

$$\nabla^2 \Upsilon = \frac{1}{c^2} \cdot \frac{\partial^2 \Upsilon}{\partial t^2} \quad (38)$$

is given by [3] traveling plane waves $\Upsilon = \Upsilon(ct \pm nr)$, where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

Υ is a smooth function representing a particular solution to equation (38), c is the speed, t is time, $\vec{n}$ is a unit vector in the direction of wave propagation, and $nr = n_x x + n_y y + n_z z$, where n_x, n_y, n_z are the direction cosines of the angles between the direction of wave propagation and the corresponding coordinate axes. It should be noted that in acoustics, it is convenient to associate the function Υ with the Heaviside step function. If an instantaneous point perturbation (e.g., an impact) occurs in the medium at the initial time ($t=0$) the initial condition is conveniently defined via the Heaviside step function, which "switches on" step-wise upon the arrival of the wave front moving at speed c . Furthermore, the function Υ can be regarded as the pressure exerted by the plane wave on an obstacle, and its relation to the potential $\Psi(r, \theta, \varphi, t)$ can be expressed as follows:

$$\Upsilon = -\rho \frac{\partial \Psi}{\partial t}, \quad (39)$$

where ρ is the material density.

Transforming from Cartesian to spherical coordinates in the expression for the function Υ , applying the forward Laplace transform (8) to the result, and using boundary condition (2), we obtain the following:

$$\left. \frac{\partial \Psi_{diff}}{\partial r} \right|_{\substack{r=R_+ \\ \varphi=-\pi, \pi}} = \Psi'_p(R+0, \theta, \varphi) = \frac{\sin \theta}{p} (\exp_p^+ - \exp_p^-) = f_p \Big|_{r=R}, \quad (40)$$

where

$$\begin{aligned} \exp_p^+ &= \cos(\varphi + \alpha) \cdot \exp \left[-p(R - R \sin \theta \cos(\varphi + \alpha)) \right], \\ \exp_p^- &= \cos(\varphi - \alpha) \cdot \exp \left[-p(R - R \sin \theta \cos(\varphi - \alpha)) \right], \end{aligned}$$

and α is the frontal inclination angle of the shock wave.

Substituting f_p from (40) into equation (34), after formally replacing the angles θ with ϑ , and φ with ψ , we obtain the final expression for the solution to the formulated problem.

It should be noted that boundary condition (3) for the function f_p is satisfied automatically.

4.4.1 Justification for Choosing the Heaviside Step Function Model for a Shock Wave

The Heaviside step function is convenient for describing a plane shock wave because it represents an almost ideal (albeit simplified) jump, characterized by the following: ahead of the wavefront, the parameters of the medium are equal to zero or to a baseline background state present in the medium, while behind the wavefront, they change instantaneously and remain constant. The value of the Heaviside function is zero ahead of the front and unity behind it, which precisely mimics the sharp (discontinuous) rise in pressure or density in an ideal shock wave.

Furthermore, such a mathematical step function facilitates the formulation and satisfaction of boundary and initial conditions for the equations of continuum mechanics, which is a highly advantageous factor; its derivative, in turn, yields the Dirac delta function, enabling the description of instantaneous force impulses that generate the wave itself.

As for the wavefront itself, it should be understood as a surface separating the disturbed region of the medium from the undisturbed one, across which pressure, density, and velocity undergo an abrupt change. The shock wave itself should be perceived as a powerful compression front within which the medium parameters increase jump-wise.

Notably, Heaviside originally applied his function exclusively to transient processes in electrical circuits; however, in gas dynamics and shock wave theory, this mathematical abstraction was adopted later by theoretical physics schools worldwide (particularly in works on the Riemann problem / decay of an arbitrary discontinuity and internal wavefronts).

4.4.2 Main Fields of Application of the Heaviside Function

Problems involving the diffraction of a step-like shock wave (described by the Heaviside function) on a sphere are applied in aerodynamics, geophysics, and protective technologies for calculating dynamic loads, evaluating structural integrity, and modeling blast waves.

For instance, in the aerospace and defense sectors, this includes calculating peak loads and pressure distribution on the nose cones of missiles, spacecraft, and military equipment upon interaction with supersonic blast waves.

In industry, it is used for safety assessments of impact loading caused by explosions on cylindrical or spherical tanks, gas holders, and containment structures of nuclear power plants.

In geophysics and acoustics, it involves modeling the propagation and scattering of seismic or hydroacoustic pulses by spherical inhomogeneities in the Earth's crust and the ocean floor.

In materials science, it serves to study transient stress states in spherical structural elements (such as bearings or capsules) under impulsive mechanical loading.

5 DISCUSSION OF RESEARCH RESULTS

Based on the results presented in Section 4, the following conclusions regarding the conducted study can be drawn:

1. The main objective of this work has been achieved, namely, demonstrating the applicability of transferring the discontinuous solution method from the theory of elasticity [11] to the classical problem of plane shock wave diffraction in an elastic space, as well as establishing the complete theoretical foundation for future numerical investigations.

2. A discontinuous solution to the wave equation in spherical coordinates (34) satisfying all boundary conditions has been constructed.

3. The problem of shock wave diffraction (38) in an elastic space by a hemispherical object bonded to the seabed (or bottom) has been formulated and solved.

4. The convergence of the obtained solution was verified to enable the calculation of the diffracted field both within the defect itself and at a distance, which will be the subject of further (numerical and graphical) analysis to be presented in a future paper.

5. The obtained result can be assessed as a novel approach to investigating the steady-state shock wave diffraction problem on spherical defects, based on the application of the discontinuous solution method and the generalized integral transform.

6. By additionally applying the inverse Laplace transform to (34), restoring the time variable t , and accounting for the initial conditions with respect to this variable, one can proceed to a broader class of problems, namely transient (non-stationary) problems in the theory of elasticity.

7. By varying the parameters in formula (26) for the wavenumber q , as well as the values of the spherical coordinates (r, θ, φ) , one can calculate the value of the potential $\Psi_p(r, \theta, \varphi)$ at any point in space for a wide range of materials, reproducing the complete diffraction pattern arising in the elastic space.

8. The constructed discontinuous solutions of the wave and three-dimensional equations of linear elasticity theory, applied to the problem of torsional wave diffraction by rigid stationary (movable) shell-type spherical bodies [11], completely coincide (up to constant factors) with the solution (34) obtained in the present problem. Moreover, relying on the results of [11], a torsional wave can be used as an alternative to the shock wave under consideration, provided all boundary conditions of the above problem are satisfied.

6 CONCLUSIONS

In accordance with the set objectives (Section 3) and the obtained results (Section 5), a universal solution to the wave equation in spherical coordinates was derived using the discontinuous solution method. This solution allows application to a wide spectrum of

acoustic problems, covering both steady-state and transient problems in elasticity theory with spherical inclusions of various physical natures.

7 ETHICAL DECLARATIONS

The authors have no relevant financial or non-financial interests to disclose.

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