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ON THE NUMBER OF INDEPENDENT COMPONENTS OF A SKEW-SYMMETRIC TENSOR

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Abstract: This paper investigates a relevant problem in modern differential geometry concerning the exact estimation of the number of independent and dependent components of a twice-covariant tensor. The study is conducted within the framework of a multidimensional pseudo-Riemannian space of dimension four or higher, endowed with a given non-degenerate metric tensor. Primary attention is given to the detailed analysis of the properties of a skew-symmetric tensor satisfying a specific tensor equation that involves an affinor. This affinor is functionally independent of the tensor under consideration, yet it satisfies a strict skew-symmetry condition with respect to the underlying metric of the space.

To rigorously prove the key mathematical propositions, a method of linear non-degenerate coordinate transformation at a fixed point of the manifold is applied, allowing the affinor in question to be algebraically reduced to its Jordan canonical form. The paper provides a rigorous proof of a lemma establishing the explicit analytical expression of certain components of the skew-symmetric tensor in terms of its other components and the entries of the affinor, provided that the sum of the corresponding eigenvalues of the affinor matrix is non-zero.

Furthermore, the calculations reveal a significant structural constraint: if the number of dependent tensor components does not exceed a specific threshold value, which is linearly dependent on the total dimension of the space, then at most two eigenvalues in the spectrum of the affinor matrix can be non-zero.

As the main theoretical result, a theorem is proven stating that if the rank of the affinor matrix exceeds two, the total number of dependent components of the tensor is guaranteed to be no less than the aforementioned threshold value. Additionally, the article considers a specific geometric case where the given affinor is constructed directly from the components of a non-zero Riemann curvature tensor using two linearly independent vector fields.

The obtained mathematical results and established regularities may be beneficial for future research in differential geometry, general relativity, and theoretical mechanics, particularly in the complex modeling of multidimensional physical processes.

Keywords: pseudo-Riemannian spaces; affinor structure; Jordan canonical form; skew-symmetric tensor; affinor-semisymmetric spaces.

ПРО КІЛЬКІСТЬ НЕЗАЛЕЖНИХ КОМПОНЕНТ КОСОСИМЕТРИЧНОГО ТЕНЗОРА

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Анотація: У представлений науковій статті досліджуються актуальна проблема сучасної диференціальної геометрії, яка пов'язана з точною оцінкою кількості незалежних та залежних компонент двічі коваріантного тензора. Дослідження проводиться для багатовимірного псевдоріманового простору, де розмірність многовиду становить чотири або більше, із заздалегідь заданим не виродженим метричним тензором. Головна увага приділена детальному вивченню властивостей кососиметричного тензора, що задовольняє спеціальному тензорному рівнянню за участі афінора. Цей афінор є функціонально незалежним від самого досліджуваного тензора, проте відповідає строгій умові косої симетрії відносно базової метрики даного простору.

Для послідовного доведення ключових математичних тверджень успішно застосовується метод лінійного невиродженого перетворення локальних координат у спеціально обраній фіксованій точці многовиду, завдяки чому досліджуваний афінор алгебраїчно приводиться до своєї канонічної форми Жордана. У роботі строго доведено лему про принципову можливість явного аналітичного вираження одних компонент кососиметричного тензора через інші його компоненти та елементи афінора за умови, що сума відповідних власних чисел матриці цього афінора не дорівнює нулю.

Крім того, в результаті обчислень встановлено важливе структурне обмеження: якщо кількість залежних компонент тензора не перевищує спеціально визначеного порогового значення, яке знаходиться в прямій лінійній залежності від загальної розмірності простору, то серед усього спектру власних чисел матриці афінора не більше двох можуть бути відмінними від нуля.

Як основний фундаментальний теоретичний результат доведено теорему про те, що у випадку, коли ранг матриці афінора перевищує два, загальна кількість залежних компонент тензора гарантовано є не меншою за вказане порогове значення. Також у статті окремо розглядається специфічний геометричний випадок, коли зазначений афінор конструюється безпосередньо з компонент ненульового тензора кривини Рімана за допомогою використання двох лінійно незалежних векторних полів.

Отримані математичні результати та виведені закономірності можуть бути корисними для подальших перспективних досліджень у галузі диференціальної геометрії, загальної теорії відносності та теоретичної механіки при комплексному моделюванні багатовимірних фізичних процесів.

Ключові слова: псевдоріманові простори; афінорна структура; канонічна форма Жордана; кососиметричний тензор; афінор-напівсиметричні простори.

1 INTRODUCTION

Consider C_n a complex-analytic manifold [1] of dimension n . Let $z^\alpha = x^\alpha + ix^{\bar{\alpha}}$ be local complex coordinates, where $\alpha = 1, 2, \dots, n$, and $\bar{\alpha} = n+1, n+2, \dots, 2n$. The numbers x^α and $x^{\bar{\alpha}}$ will be considered as local coordinates of a real even-dimensional analytic manifold M_{2n} .

If R_z – the tangent space to the manifold C_n at the point z [2], T_x – the tangent space to M_{2n} at the point x . The linear transformation F is defined on M_{2n} by a tensor (affinor) field with local components F_j^i . In general, spaces in which an affinor field satisfying certain conditions is defined are called spaces with an affinor structure, and the affinor itself, which is subjected to the conditions, is called a structural affinor [3]. In the given space M_{2n} , this field (or affinor) satisfies the condition

$$F_\alpha^i F_j^\alpha = -\delta_j^i, \\ (i, j, \alpha = 1, 2, \dots, 2n),$$

where δ_i^h – Kronecker deltas, which are defined by the following conditions

$$\delta_i^h = \begin{cases} 1, & i = h \\ 0, & i \neq h \end{cases}.$$

The tensor field F_j^i – is called an almost complex structure on M_{2n} generated by a complex analytic structure on C_n .

More general cases are also considered in the literature. Thus, for example, a structure F_j^i such that $F_\alpha^h F_j^\alpha = e\delta_i^h$ is called an *e-structure* of *elliptic type* for $e = -1$, of *parabolic type* for $e = 0$, a structure of *hyperbolic type* – for $e = 1$.

If a tensor F is given in the tangent space T_x to the manifold, then the complex extension of this space T_x^c decomposes into a direct sum of subspaces S_n^c and $\bar{S}_n^c$ – linear spans of eigenvectors belonging to the eigenvalues i and $-i$, respectively. Such a decomposition of T^c into a direct sum is sometimes used to define spaces with an almost complex structure [4].

An almost complex structure determines on the manifold fields of two-dimensional directions called two-dimensional analytic directions or characteristic planes. A two-dimensional analytic direction in T_x passing through a vector v is called the linear span of the vectors v and Fv . On a characteristic plane, an almost complex structure determines an orientation, an angular metric, and a relation of equality of vector lengths. A system of oriented characteristic planes determines an almost complex structure.

An almost complex structure does not exist on every even-dimensional orientable manifold of class C^∞ . An example of such a manifold that does not admit an almost complex structure is the four-dimensional and eight-dimensional spheres. For an almost complex structure to exist on a manifold, it is necessary and sufficient that there exists on it an exterior differential form of the second order ω such that $\Lambda^n \omega \neq 0$ or, in other words, the rank of the matrix of local components of the form ω is equal to $2n$. On the other hand, the presence of an almost complex structure on a manifold is not yet sufficient to allow the introduction of the structure of a complex-analytic manifold on it.

An almost Hermitian structure is defined on a manifold with an almost complex structure by specifying a metric tensor that satisfies the condition

$$g_{\alpha\beta} F_i^\alpha F_j^\beta = g_{ij}. \quad (1)$$

This condition expresses the fact that the inner product of vectors remains invariant under the transformation F . An almost Hermitian structure can be introduced on any space with an almost complex structure. The tensor $F_{ij} = g_{ik}F_j^k$ defines an exterior differential 2-form F of rank $2n$ on the manifold as a whole. An almost Hermitian structure can be defined on a Riemannian manifold by specifying an exterior differential 2-form of rank $2n$ that satisfies the condition (1).

By imposing algebraic or differential conditions on a Hermitian structure, special types of spaces are obtained. One of the well-known types of spaces is Kähler spaces, in which the almost Hermitian structure is covariantly constant (or absolutely parallel) with respect to the connection of the space M_{2n} , that is, $F_{ij,k} = 0$.

A direct generalization of Kähler spaces is affino-semisymmetric spaces.

Definition. A space V_n is called *affino-semisymmetric* if a semisymmetric affino structure $F_i^h(x)$ is defined in it, that is, the affino $F_i^h(x)$ satisfies the conditions $F_{ij,[kl]} = 0$.

Here the index is lowered using the metric tensor $F_{ij} = g_{i\alpha}F_j^\alpha$. The structure of such a space will be called *F-semisymmetric*.

Intrinsic objects of the space V_n in which an affino structure is defined include objects determined from the metric tensor g_{ij} and the structural affino F_i^h .

2 ANALYSIS OF LITERATURE AND PROBLEM STATEMENT

The specialization of pseudo-Riemannian spaces, as a means of introducing additional constraints, has two main sources. The first is the imposition of geometric conditions on the object under study and the utilization of geometric features in the investigation. The second consists of technological capabilities that arise when using developed research methods. Taking both features into account, methods for studying tensor fields that satisfy special relations with affino structures have been proposed.

Among pseudo-Riemannian spaces that admit the existence of additional structures, affino-semisymmetric spaces are highlighted, where the affino is constructed directly from the Riemann tensor. Their algebraic features are studied, and based on this, the number of independent components of intrinsic tensor objects is evaluated.

Spaces of dimension $n \geq 4$ have their own peculiarities both in research methods and in the obtained results. Technical difficulties that arise in the process of studying systems of differential and tensor equations have a specific character, while the methods for overcoming them by reducing the affino to the Jordan canonical form the basis of local analysis. Therefore, the study of dependencies between the components of twice-covariant tensors is a relevant topic of research. The work is carried out locally, using tensor methods.

3 AIM AND OBJECTIVES OF THE STUDY

The study of algebraic and tensor relations is one of the effective methods for studying intrinsic geometry in differential geometry. In our case, this provides an opportunity to evaluate the number of independent and dependent components of a twice-covariant skew-symmetric tensor in a pseudo-Riemannian space by reducing an independent affino to a special form – the Jordan canonical form in some coordinate system.

The aim of the work is to establish an exact criterion for the dependence of the components of a skew-symmetric tensor on its other components and elements of the affino, as well as to determine the relationship between the rank of the affino matrix and the minimum number of dependent components for space dimension $n \geq 4$. A separate task is to apply the obtained results to affino-semisymmetric spaces, where the affino structure is

directly generated by the Riemann tensor, which is important for broad classes of pseudo-Riemannian spaces from general relativity, theoretical mechanics, as well as other branches of differential geometry and topology.

4 RESEARCH RESULTS

Consider a pseudo-Riemannian space V_n [5] with a metric tensor $g_{ij}(x)$ in which a twice-covariant skew-symmetric tensor F_{ij} ($F_{ij} = -F_{ji}$) is defined.

Let us estimate the number of independent components [6] of the tensor F_{ij} satisfying the equation: $F_{\alpha j} K_{\cdot i}^{\alpha} + F_{i\alpha} K_{\cdot j}^{\alpha} = 0$.

Here K_j^h is an affinor independent of F_{ij} . Additionally, we assume that it satisfies the skew-symmetry condition [4] with respect to the metric: $g_{\alpha i} K_j^{\alpha} + g_{\alpha j} K_i^{\alpha} = 0$.

Consider a linear non-degenerate coordinate transformation at some fixed point $M(x^h) \in V_n$: $y^h = A_{\alpha}^h x^{\alpha}$, where $\det \|A_i^h\| \neq 0$.

The tensors F_{ij} and K_h^i transform according to the law [6, 7]:

$$F'_{ij} = F_{\alpha\beta} B_i^{\alpha} B_j^{\beta}, \quad K_i'^h = K_{\beta}^{\alpha} B_i^{\beta} A_{\alpha}^h, \quad \text{here } \|B_i^h\| = \|A_i^h\|^{-1}.$$

Obviously, the rank of the tensor K_i^h , as well as the number of dependent components of the tensor F_{ij} in our equation, are invariant with respect to coordinate transformations. The latter remains true even when $\|A_i^h\|$ is a complex-valued non-degenerate matrix.

For further investigation, we reduce the affinor K_i^h using a non-degenerate transformation (the case of a complex transformation is not excluded) to the Jordan canonical form. In our study, we restrict ourselves to a simplified notation of this matrix:

$$K_j^i = \begin{pmatrix} \omega_1 & K_1^2 & 0 & \dots & 0 \\ 0 & \omega_2 & K_2^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega_n \end{pmatrix}.$$

The latter can be written as follows: $K_{\alpha}^{\alpha} = \omega_{\alpha}$, $K_{\alpha}^{\alpha+1} = 0, 1$, $K_b^a = 0$ for $b \neq a, a+1$. In this formula, as throughout the paper, summation is not implied over repeated indices.

Naturally, in the general case, the eigenvalues ω_{α} of the matrix K_i^h can be complex. The coordinate system y in which K_i^h has the Jordan form will be called canonical.

The equation of the original system with fixed indices i and j will be denoted by θ_{ij} . Since the equation θ_{ji} coincides up to a sign with the equation θ_{ij} (due to the property $F_{ij} = -F_{ji}$), our system is equivalent to the system of equations for indices $i < j$. Diagonal equations θ_{ii} hold identically.

The equations in the canonical coordinate system y have the following form:

$$\theta_{ij} (i < j < n): \quad F_{ij}(\omega_i + \omega_j) + F_{i+1,j} K_i^{i+1} + F_{i,j+1} K_j^{j+1} = 0;$$

$$\theta_{in} (i < n): \quad F_{in}(\omega_i + \omega_n) + F_{i+1,n} K_i^{i+1} = 0.$$

Recall that the Einstein summation convention does not apply to these equations. We prove the following lemma:

Lemma 1.

The components of the skew-symmetric tensor F_{ij} (where $i < j$) for which $(\omega_i + \omega_j) \neq 0$ depend on the other components of the tensor F_{ij} and the affnor K_j^i .

Proof. Let I denote the set of ordered pairs of indices (i, j) , where $i < j$, $i, j = 1, 2, \dots, n$. Let I_1 and I^* denote subsets such that:

$$(i, j) \in I_1 \Leftrightarrow (\omega_i + \omega_j) \neq 0 \quad \text{and} \quad I^* = I_1.$$

Let q denote the number of elements in I_1 . Below we will prove that q components F_{ij} , where $(i, j) \in I_1$, are explicitly expressed in terms of other components of the tensor F_{ij} and the affnor K_j^h .

Step 1. From the set of pairs I_1 , select a pair (i_1, j_1) such that the indices i_1 and j_1 are "maximal", that is:

$$j_1 = \max_{\forall (i, j) \in I_1} j \quad \text{and} \quad i_1 = \max_{\forall (i, j_1) \in I_1} i.$$

By analyzing the equation $\theta_{i_1 j_1}$, one can explicitly express the component $F_{i_1 j_1}$ in terms of the components $F_{ij} \in I^*$ and K_i^h . This follows from the fact that the indices in the terms F_{i_1+1, j_1} and F_{i_1, j_1+1} are strictly larger; therefore, these components are either identically zero or already belong to the set I^* .

Step 2. Next, let us denote $I_2 = I_1 \setminus \{(i_1, j_1)\}$. Similarly, from the set of pairs I_2 , we select a "maximal" pair (i_2, j_2) :

$$j_2 = \max_{\forall (i, j) \in I_2} j \quad \text{and} \quad i_2 = \max_{\forall (i, j_2) \in I_2} i.$$

Then, by analogy with the previous step, it is easy to see that by analyzing the equation $\theta_{i_2 j_2}$, one can explicitly express the component $F_{i_2 j_2}$ in terms of the component $F_{i_1 j_1}$ and the components F_{ij} for $(i, j) \in I^*$ and K_i^h . But then, based on Step 1, it follows that $F_{i_2 j_2}$ is also expressed explicitly only in terms of the components from I^* and K_i^h .

We continue this process similarly. After q steps, we make sure that all components F_{ij} for which $(i, j) \in I_1$ are explicitly expressed only through the components F_{ij} for $(i, j) \in I^*$ and the affnor K_i^h .

The lemma is proved.

Based on Lemma 1, we can formulate and prove the following lemma. Let r^* be a number equal to $2n - 5$ for pseudo-Riemannian spaces V_n with $n \geq 4$.

Lemma 2.

If the number of dependent components of the tensor F_{ij} does not exceed the number $r^* = 2n - 5$, then among the numbers ω_i there are no more than two non-zero ones.

Proof. We carry out the proof by contradiction. Assume that there are at least three non-zero eigenvalues $\omega_a, \omega_b, \omega_c$. Since the indices can be renumbered, we will assume that $a=1, b=2, c=3$.

Let us show that under this assumption at least $2n - 5$ components of the tensor F_{ij} depend on other components of this tensor and the affnor K_i^h . Taking into account Lemma 1, for this it is sufficient to show that among the elements $\omega_i + \omega_j$ (where $i < j$) there are at least r^* non-zero ones.

1. First, consider the case where the non-zero eigenvalues satisfy the condition:

$$|\omega_1| \neq |\omega_2| \neq |\omega_3|.$$

Then it is obvious that among the first three indices, three sums are non-zero:

$$\omega_1 + \omega_2, \quad \omega_1 + \omega_3, \quad \omega_2 + \omega_3.$$

For any other index $a > 3$:

- **if $\omega_a = 0$, then $\omega_1 + \omega_a$, $\omega_2 + \omega_a$, $\omega_3 + \omega_a$, are non-zero (3 pairs);**
- **if $\omega_a = \pm\omega_1$, then at least $\omega_2 + \omega_a$ and $\omega_3 + \omega_a$, are non-zero (2 pairs);**
- **if $\omega_a = \pm\omega_2$ or $\pm\omega_3$, we similarly have at least 2 non-zero pairs.**

Hence, based on Lemma 1, it follows that among the components of F_{ij} there are at least $3 + 2(n-3) = 2n-3$ dependent ones, which contradicts the condition $r^* \leq 2n-5$.

2. Consider the case where the eigenvalues satisfy the condition:

$$\omega_1 = -\omega_2, \quad \omega_3 \neq \pm\omega_1,$$

and for $\forall a > 3$: $\omega_a = \pm\omega_1, \pm\omega_3, 0$. (Other values reduce to the previous case).

Among the first three indices, two sums are non-zero: $\omega_1 + \omega_3$ and $\omega_2 + \omega_3$.

For $a > 3$:

- **if $\omega_a = 0$, we have 3 non-zero sums;**
- **if $\omega_a = \omega_1$, $\omega_1 + \omega_a = 2\omega_1$ and $\omega_3 + \omega_a$ are non-zero (2 pairs);**
- **if $\omega_a = -\omega_1$, $\omega_2 + \omega_a = -2\omega_1$ and $\omega_3 + \omega_a$ are non-zero (2 pairs).**

Thus, the number of dependent components is not less than $2 + 2(n-3) = 2n-4 > 2n-5$. Again, a contradiction.

3. If $\omega_1 = \omega_2$ and $\omega_3 \neq \pm\omega_1$.

Here, among the first three we have $\omega_1 + \omega_2 = 2\omega_1 \neq 0$, $\omega_1 + \omega_3 \neq 0$, $\omega_2 + \omega_3 \neq 0$ (3 pairs). Together with at least 2 pairs for each $a > 3$, we obtain $3 + 2(n-3) = 2n-3$ dependent components.

4. Obviously, it remains to consider the case where:

$$\omega_1 = \dots = \omega_k = -\omega_{k+1} = \dots = -\omega_{k+l} \neq 0, \quad \omega_{k+l+1} = \dots = \omega_n = 0.$$

The number of non-zero sums $\omega_i + \omega_j$ ($i < j$) in this case is equal to:

$$S = \frac{k(k-1)}{2} + \frac{l(l-1)}{2} + (k+l)(n-k-l).$$

Since by assumption $k+1 \geq 3$, the minimum value of S is reached, for example, at $k = 2$, $l = 1$. Then $S = 1 + 0 + 3(n-3) = 3n-8$. It is easy to verify that for $n \geq 4$, $3n-8 \geq 2n-5$ holds.

As a result, we have considered all possible combinations of eigenvalues and obtained a contradiction.

Lemma 2 is proved.

Based on the previous lemmas, we prove

Theorem.

If the rank of the matrix $\|K_j^i\|$ is greater than two, then among the components of the tensor F_{ij} at least r^* depend on other components of the tensor F_{ij} and the affiner K_j^i .

Proof. It is sufficient for us to show that if the number of dependent components of the tensor F_{ij} is less than or equal to $2n - 5$, then the matrix of the tensor K_j^i has no more than two linearly independent rows.

We carry out the proof by contradiction. In the following, we assume that in the matrix K_j^i reduced to the Jordan form, there are at least three non-zero rows.

Taking into account Lemma 2 (no more than two non-zero diagonal elements) and the possibility of permuting Jordan blocks, it is sufficient to consider the following cases for the structure of the super diagonal elements $K_j^{i+1} = \varepsilon_j$:

Case 1. $\omega_1, \omega_2 \neq 0$, $\omega_a = 0$, for $a = \overline{3, n}$, and $K_3^4 = 1$.

It is obvious that the numbers $\omega_1 + \omega_a$ and $\omega_2 + \omega_a$ ($\forall a \geq 3$) are non-zero. By Lemma 1, the components F_{1a} ($a \geq 3$) and F_{2a} ($a \geq 3$) depend on other components.

The number of these dependent components is $2(n-2) = 2n - 4$. Since $2n - 4 > 2n - 5$, we have a contradiction.

Case 2. $\omega_1 \neq 0$, $\omega_a = 0$ for $a = \overline{2, n}$ and $K_2^3 = K_3^4 = 1$.

Since $\omega_1 + \omega_a \neq 0$ ($a = \overline{2, n}$), by Lemma 1 the components F_{1a} ($n - 1$ of them) are expressed in terms of the others.

From the equation θ_{in} for $i = 2, 3$ it follows that $F_{3n} = F_{4n} = 0$.

The formula θ_{ij} for $i = 2, j = s$ ($s = \overline{3, n-1}$) yields: $F_{3s} = -F_{2, s+1} K_s^{s+1}$.

Thus, additionally F_{3a} ($a = 4, 5, \dots, n$) depend on other components ($n - 3$ components in total).

The total number of dependent components is: $(n-1) + (n-3) = 2n - 4 \geq 2n - 5$. Again, a contradiction.

Case 3. $\omega_1 \neq 0$, $\omega_a = 0$ for $a = \overline{2, n}$, $K_2^3 = 1, K_3^4 = 0, K_4^5 = 1$.

Similarly to the second case, F_{1a} ($n-1$ of them) are expressed in terms of the others.

The equation θ_{2s} gives a dependence for F_{3a} , and the equation θ_{4s} — for F_{5a} . After careful counting, we see that the number of dependent components strictly exceeds $2n - 5$.

Other cases. (All $\omega_a = 0$, but there are chains of ones, for example $K_1^2 = K_2^3 = K_3^4 = 1$).

In these cases, the system θ_{ij} reduces to $F_{i+1, j} \varepsilon_i + F_{i, j+1} \varepsilon_j = 0$.

Analysis of these recurrence relations leads to the fact that the number of zero and linearly dependent diagonals of the matrix F_{ij} generates at least $2n - 5$ dependent components.

Thus, in all possible cases we have obtained a contradiction. Consequently, the assumption about the existence of three linearly independent rows was false. The theorem is proved.

Let an affinor-semisymmetric structure be defined in the space, that is, a structure for which $F_{ij, [kl]} = 0$, holds, or, taking into account the integrability conditions of F_{ij} ,

$$F_{\alpha j} R_{ikl}^{\alpha} + F_{i\alpha} R_{jkl}^{\alpha} = 0.$$

In a non-trivial case, the Riemann tensor R_{ijk}^h [8] of this space is not identically zero; therefore, there exist two vector fields μ^i, ν^i [9] such that $R_{i\alpha\beta}^h \mu^{\alpha} \nu^{\beta} \neq 0$.

We introduce the notation: let

$$K_{.i}^h \stackrel{\text{def}}{=} R_{i\alpha\beta}^h \mu^{\alpha} \nu^{\beta}. \quad (2)$$

Thus, we obtain K_j^h – an affinor independent of F_{ij} , constructed from the Riemann tensor. Obviously, it satisfies the skew-symmetry condition with respect to the metric [10]:

$$g_{\alpha i} K_j^\alpha + g_{\alpha j} K_i^\alpha = 0,$$

and therefore, for these spaces the following holds:

Corollary 1.

In affinor-semisymmetric spaces, if the rank of the matrix $\|K_j^i\|$ is greater than two, then among the components of the tensor F_{ij} at least r^* depend on other components of the tensor F_{ij} and the affinor K_j^i .

5 DISCUSSION OF RESEARCH RESULTS

To verify and illustrate the results obtained in Lemma 1, Lemma 2, Theorem, and Corollary 1, let us consider a specific example of a four-dimensional ($n = 4$) non-flat affinor-semisymmetric non-Kähler pseudo-Riemannian space V_4 .

The metric tensor g_{ij} of this space is given in matrix form as:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (x^1)^2 & 0 \\ 0 & 0 & 0 & (x^1)^2 \end{pmatrix}.$$

By direct calculation, we find the Christoffel symbols [6, 10] of the second kind, which we represent as matrices with respect to the upper index Γ_{ij}^k :

$$\Gamma_{ij}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x^1 & 0 \\ 0 & 0 & 0 & -x^1 \end{pmatrix}, \quad \Gamma_{ij}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_{ij}^3 = \begin{pmatrix} 0 & 0 & \frac{1}{x^1} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{x^1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_{ij}^4 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{x^1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{x^1} & 0 & 0 & 0 \end{pmatrix}.$$

Using the obtained matrices, we compute the components of the Riemann curvature tensor R_{jkl}^i . The only non-zero components (up to skew-symmetry in the last two indices) are:

$$R_{.434}^3 = -1, \quad R_{.334}^4 = 1. \quad (3)$$

Since the Riemann tensor is not identically zero, the space is non-flat.

We specify the structural affinor F_j^i by the matrix:

$$F_{.j}^i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (4)$$

It is directly visible that $F^2 = -I$, confirming the presence of an almost complex structure. Lowering the upper index using the metric tensor ($F_{ij} = g_{i\alpha} F_{.j}^{\alpha}$), we obtain the covariant affinor in matrix form:

$$F_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (x^1)^2 \\ 0 & 0 & -(x^1)^2 & 0 \end{pmatrix}.$$

Since $F_{ij} = -F_{ji}$, the structure is skew-symmetric.

To study the differential properties, we compute the covariant derivative $\nabla_k F_{.j}^i$. Due to the constancy of the matrix (4), we have: $\nabla_k F_{.j}^i = \Gamma_{\alpha k}^i F_{.j}^{\alpha} - \Gamma_{jk}^{\alpha} F_{.\alpha}^i$.

In particular, for the component $\nabla_3 F_{.2}^3$, we obtain:

$$\nabla_3 F_{.2}^3 = \Gamma_{13}^3 F_{.2}^1 - \Gamma_{23}^m F_{.m}^3 = \frac{1}{x^1} \cdot 1 - 0 = \frac{1}{x^1} \neq 0.$$

Thus, the structure is non-Kähler.

Let us verify the semisymmetry condition, which via the Ricci identity is written as:

$$F_{ij,[kl]} = F_{\alpha j} R_{.ikl}^{\alpha} + F_{i\alpha} R_{.jkl}^{\alpha}. \quad (5)$$

Since the non-zero components of the curvature tensor are determined only by the indices 3 and 4 according to (3), it is sufficient to verify equality (5) for $k = 3, l = 4$. Consider the possible non-zero combinations for the indices i and j :

1) For $i = 3, j = 3$:

$$F_{43} R_{.334}^4 + F_{34} R_{.334}^4 = -(x^1)^2 \cdot 1 + (x^1)^2 \cdot 1 = 0,$$

2) For $i = 4, j = 4$:

$$F_{34} R_{.434}^3 + F_{43} R_{.434}^3 = (x^1)^2 \cdot (-1) + (-(x^1)^2) \cdot (-1) = 0,$$

3) For $i = 3, j = 4$:

$$F_{44} R_{.334}^4 + F_{33} R_{.434}^3 = 0 \cdot 1 + 0 \cdot (-1) = 0.$$

For all other sets of indices, the components of the curvature tensor or the covariant affinor make the corresponding terms zero. Thus, the identity $F_{ij,[kl]} = 0$ holds for the entire space, proving its semisymmetry.

In the space V_4 with the metric tensor g_{ij} , where the only non-zero components of the curvature tensor are $R_{.434}^3 = -1$ and $R_{.334}^4 = 1$, the condition of affinor semisymmetry $F_{ij,[kl]} = 0$ is satisfied. According to formula **Ошибка! Источник ссылки не найден.**, we construct

an affnor $K_{.i}^h$, independent of F_{ij} , using the Riemann tensor and two orthogonal vector fields μ^α, ν^β : $K_{.i}^h \stackrel{\text{def}}{=} R_{.i\alpha\beta}^h \mu^\alpha \nu^\beta$.

We choose vector fields with non-zero components $\mu^3 = 1$ and $\nu^4 = 1$. Then the non-zero elements of the affnor matrix $\|K_{.j}^i\|$ (where the upper index indicates the row and the lower one the column) will be: $K_{.3}^4 = R_{.334}^4 \mu^3 \nu^4 = 1$, $K_{.4}^3 = R_{.434}^3 \mu^3 \nu^4 = -1$.

The affnor matrix $K_{.j}^i$ takes the form:

$$\|K_{.j}^i\| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is obvious that the rank of this matrix is equal to two ($\text{rank}(K) = 2$), and it satisfies the skew-symmetry condition with respect to the metric $g_{\alpha i} K_j^\alpha + g_{\alpha j} K_i^\alpha = 0$.

We find the eigenvalues ω_i of the matrix $\|K_{.j}^i\|$ from the characteristic equation

$$\det(K - \omega I) = 0: \omega^2(\omega^2 + 1) = 0 \Rightarrow \omega_1 = 0, \quad \omega_2 = 0, \quad \omega_3 = i, \quad \omega_4 = -i.$$

In our case, the threshold value for the number of dependent components for $n = 4$ is $r^* = 2n - 5 = 3$. Since exactly two of the eigenvalues ω_i are non-zero ($\omega_3 = i$ and $\omega_4 = -i$), this is in complete agreement with the statement of Lemma 2 (if the number of dependent components does not exceed r^* , then no more than two eigenvalues are non-zero).

According to Lemma 1, the components of the skew-symmetric tensor F_{ij} (where $i < j$) for which the sum of the corresponding eigenvalues is non-zero ($\omega_i + \omega_j \neq 0$) depend on the other components.

Checking all pairs of indices (a, b) with $a < b$ shows that for the pairs (1, 3) and (2, 3) the sum is $\omega_{1,2} + \omega_3 = i \neq 0$, for the pairs (1, 4) and (2, 4) the sum is $\omega_{1,2} + \omega_4 = -i \neq 0$, whereas for the pairs (1, 2) and (3, 4) the sums are equal to zero.

Substituting the resulting affnor $K_{.j}^i$ into the system of equations θ_{ij} for the skew-symmetric tensor: $F_{\alpha j} K_{.i}^\alpha + F_{i\alpha} K_{.j}^\alpha = 0$.

Expanding this system for all independent pairs $i < j$, we obtain the relations

$$\theta_{13} : F_{14} = 0, \quad \theta_{14} : -F_{13} = 0, \quad \theta_{23} : F_{24} = 0, \quad \theta_{24} : -F_{23} = 0,$$

while the equations θ_{12} and θ_{34} hold identically. The status of all components of the tensor F_{ij} is given in Table 1.

Table 1.

Dependence of the components of the skew-symmetric tensor F_{ij} in the space V_4

Component F_{ij}	Sum of eigenvalues $(\omega_i + \omega_j)$	Component status according to θ_{ij}
F_{12}	0	Independent
F_{13}	$i \neq 0$	Dependent ($F_{13} = 0$)
F_{14}	$-i \neq 0$	Dependent ($F_{14} = 0$)
F_{23}	$i \neq 0$	Dependent ($F_{23} = 0$)

F_{24}	$-i \neq 0$	Dependent ($F_{24} = 0$)
F_{34}	0	Independent

The total number of independent components of an arbitrary skew-symmetric tensor in a four-dimensional space is equal to $n(n-1)/2 = 6$. As can be seen from the analysis, exactly 4 components ($F_{13}, F_{14}, F_{23}, F_{24}$) are dependent (identically zero), and 2 components (F_{12}, F_{34}) remain independent. Since the resulting number of dependent components (4) satisfies the inequality $4 \geq r^* = 3$, this confirms the lower bound established in Theorem and Corollary 1.

Based on the analysis of the intrinsic geometry and differential properties of the space under consideration, the following statement is proved:

Corollary 2.

In the given four-dimensional non-flat affnor-semisymmetric non-Kähler pseudo-Riemannian space V_4 , the structural affnor K^i_j constructed using the Riemann tensor has rank two and exactly two non-zero imaginary eigenvalues $\pm i$. In this case, the system of equations θ_{ij} has exactly 4 dependent components of the skew-symmetric tensor F_{ij} , which exceeds the threshold value $r^* = 2n-5=3$ and completely agrees with the criteria of Lemmas 1-2 and Theorem.

6 CONCLUSIONS

In this work, we succeeded in investigating the algebraic and differential-geometric properties of twice-covariant tensors in pseudo-Riemannian spaces of dimension $n \geq 4$. For a skew-symmetric tensor satisfying a special linear equation with an independent affnor, the method of reducing the affnor matrix to the Jordan canonical form was applied. It is proved that the components of the skew-symmetric tensor for which the sum of the corresponding eigenvalues of the affnor is non-zero are explicitly expressed in terms of its other components and the elements of this affnor.

A constraint on the affnor itself was also proved: if the number of dependent components does not exceed the threshold value $r^* = 2n - 5$, then among the eigenvalues of the affnor matrix no more than two are non-zero. As a central result, a theorem is established stating that in the case where the rank of the affnor matrix exceeds two, at least $2n - 5$ components of the skew-symmetric tensor are linearly dependent. The obtained criteria were successfully extended to the class of affnor-semisymmetric spaces, where the structural affnor is constructed directly from the Riemann tensor using two vector fields. A specific example of a four-dimensional non-Kähler space with the specified structure was constructed, to which the results of the research were applied.

The obtained results may find application in further research in the field of differential geometry [11], general relativity, theoretical mechanics, and mathematical modeling of physical phenomena.

7 ETHICAL DECLARATIONS

The author has no relevant financial or non-financial interests to report.

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